On Pure and (approximate) Strong Equilibria of Facility Location Games

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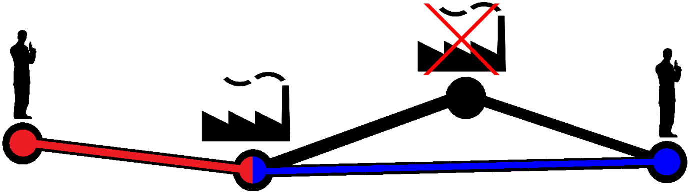
Facility Location with Fair Cost Allocation

- Set A of n agents on the nodes of a network $G(V, E, d, \beta)$
  connection cost: $d : E \rightarrow \mathbb{R}^+$, *facility installation cost:* $\beta : V \rightarrow \mathbb{R}^+$
- Agent $i \in A$ is located at u_i and has a demand w_i for service
- Agent i orders facility installation at $s_i = v \in V$
- Connects to s_i at cost $w_i \cdot d(u_i, s_i)$
- Pays a fair share for facility installation cost: $\frac{w_i \beta_v}{W_s(v)}$



Context

- The unweighted FL game is a *potential* game (Monderer, Shapley, 1996) and therefore has pure equilibria. Potential function:

$$\Phi(s) = \sum_{v \in F_s} \beta_v \cdot H(|A_s(v)|) + \sum_i d(u_i, s_i)$$

$$i \in A \text{ deviates:} \quad \Phi(s) - \Phi(s_{-i}, s_i') = c_i(s) - c_i(s_{-i}, s_i')$$

- Specializes *Selfish Network Design with Fair Cost Allocation* (Anshelevich, Dasgupta, Kleinberg, Tardos, Wexler, Roughgarden, 2004)

- Model similar to that of Chun, Chaudhuri, Wee, Barreno, Papadimitriou, Kubiatowicz, 2004, except they only allowed agents to “cache” locally or “connect” without sharing the cost

Concepts

- **We study the social cost of pure and strong equilibria:**
  - Social cost:

$$c(s) = \sum_i c_i(s) = \sum_{v \in F_s} \beta_v + \sum_i w_i d(u_i, s_i)$$

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  - α-approximate strong equilibrium:

$$\forall I \subseteq A \ \forall s_I' \in V^{|I|} \ \forall i \in I : \frac{c_i(s_{-I}, s_I')}{c_i(s)} \leq \alpha$$

Prices of Stability/Anarchy

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- Anshelevich et al. 2004: $PoS = O(\ln n)$ for unweighted FL (potential function arguments)
- Albers 2008: $SPoA_\alpha = O(\alpha \ln n)$ for network design games with fair cost sharing

Results for Facility Location

	Metric	Non-metric	
	Unweighted	Unweighted	Weighted
Existence of PNE	YES†		OPEN
PoS	$\in [1.77, 2.36]$	$\Theta(\ln n)$‡	OPEN
Existence of SE	NO		
Existence of α-approximate SE, for $\alpha \geq$	2.36	e	
$SPoA_\alpha$, with α being the above	$O(1)$	$\Theta(\ln n)$	$O(\ln W)$

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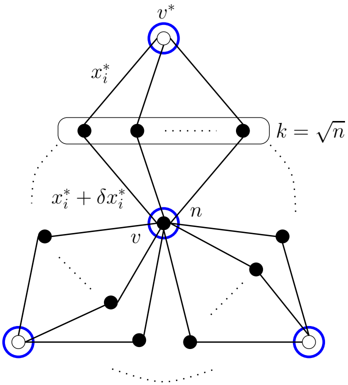
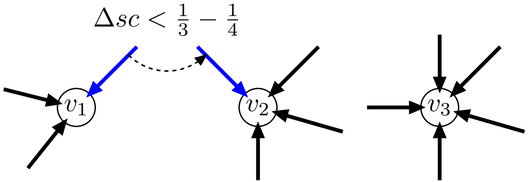
Lower Bound for *PoS* (Metric, Unweighted)

- $2n$ agents (on black nodes)
- $k = \sqrt{n}$ per *"diamond"*, n on v
- OPT: Facilities on v and v^*
- EQ: All agents play v (center)



Upper Bound for *PoS* (Metric, Unweighted)

- We bound the increase in social cost from a given configuration (e.g. social optimum) through a sequence of selfish improvements.



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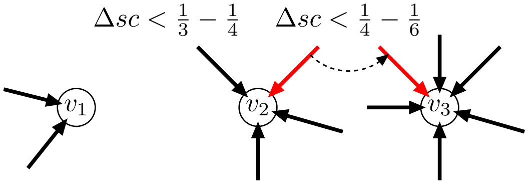

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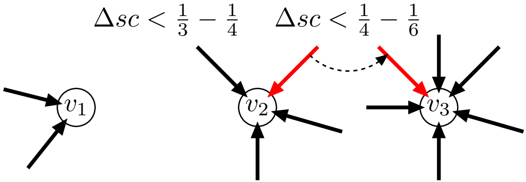

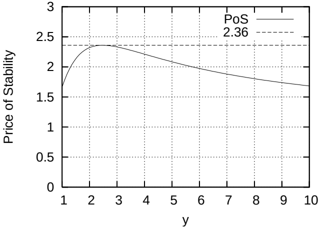

- Charging scheme + triangle inequality + calculations:

$$PoS \leq \frac{1.5 + y + \ln y}{0.5 + y - \ln y} \leq 2.36, \quad y \geq 1$$

Existence of e-approximate Strong Equilibria

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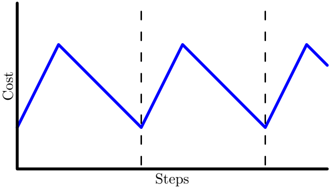

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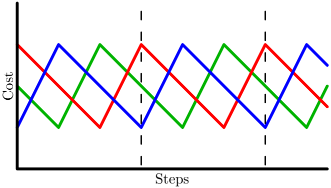

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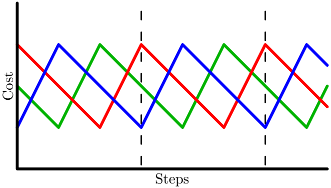

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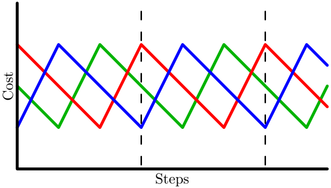

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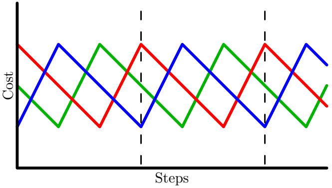

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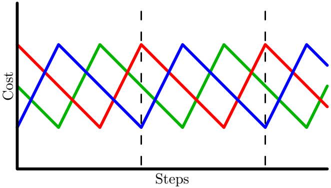

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$\vdots$	$\vdots$
$n \to \infty$	$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

Open problems

- Do pure equilibria exist for weighted agents? What is their *PoS*?
- Lower bound for the apx ratios of SE
- Computation of (approximate) equilibria in polynomial time

The end

Thank you!