On Acyclicity of Games with Cycles

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Introduction

We study games with cycles. We restrict our attention to:

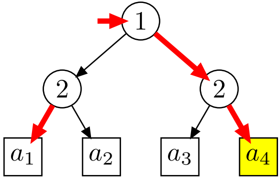
- Finite games
- Perfect information
- Deterministic
- Stationary strategies
- Pure strategies

The two-player zero-sum case (like chess) was studied by Zermelo as early as in 1912.

Solution concept: Nash equilibrium
Stronger concept: Improvement acyclicity

Positional games

- Directed graph $G = (V, E)$.
- A **starting position** $v_0 \in V$.
- Players control vertices.
- **Strategy:** Choice of **move** at every vertex.
- **Strategy profile:** A strategy for every player.



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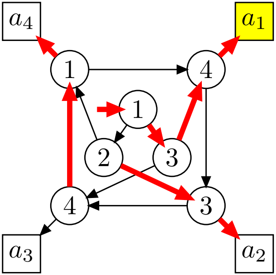
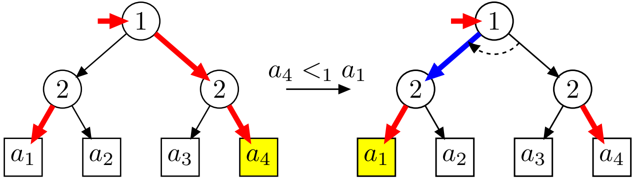


- **Play:** The path from v_0 following moves.
- **Outcomes:** The set of **terminals**.
- **Infinite play:** An extra outcome c corresponding to cycles.
- An ordering over outcomes for every player, induced by a **utility function**.

Improvement

Improvement: One player changes strategy improving his utility, while the other players' strategies stay fixed.



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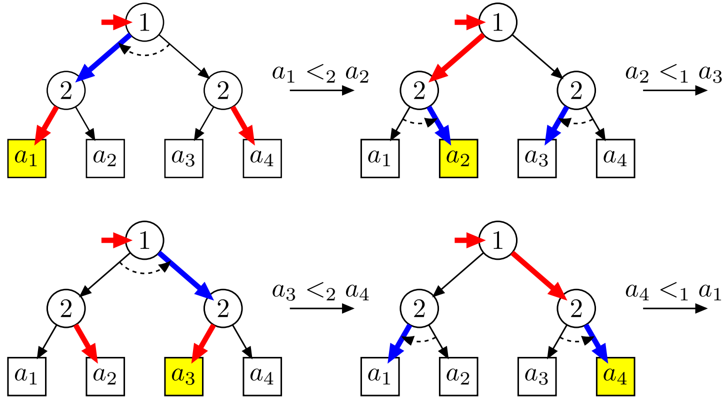
$$\begin{array}{c} \rightarrow 2 \\ \downarrow \\ 1 \end{array} \begin{bmatrix} a_1 & a_1 & a_2 & a_2 \\ & \uparrow & & \\ a_3 & a_4 & a_3 & a_4 \end{bmatrix}$$

Improvement cycle



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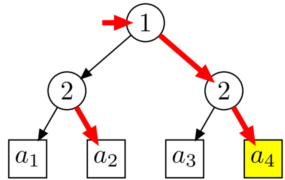
Nash equilibrium



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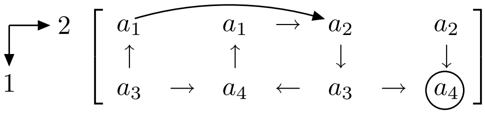
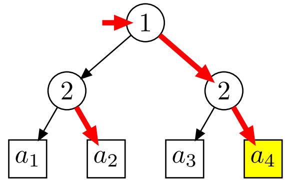


- **Nash equilibrium:** No player can improve.
- Improvement acyclicity implies existence of Nash equilibria.
- The set of possible improvements can be **restricted**. First considered by Kukushkin (2002).



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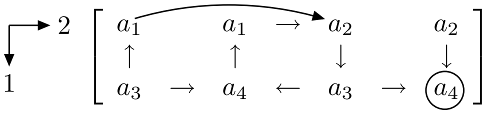

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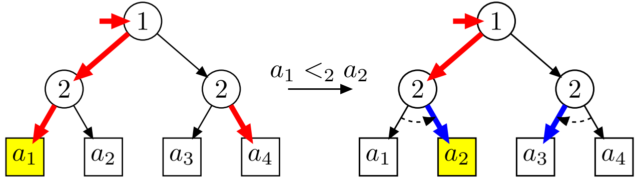
Main question: Existence of Nash equilibria of positional games with three or more players? Is **restricted improvement acyclicity** a good approach?

Restricted improvement (Kukushkin (2002))**:** Only *relevant* changes are allowed, i.e. only along the resulting play.



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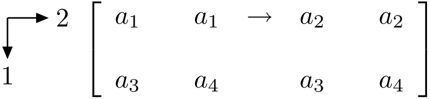

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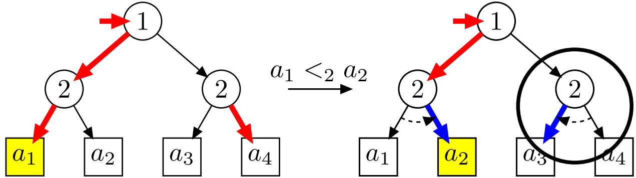

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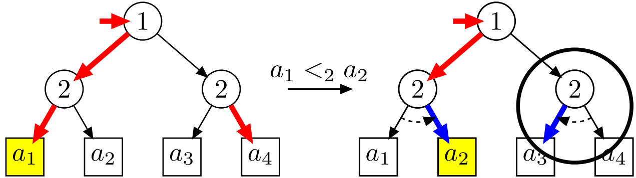

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Kukushkin (2002): Restricted improvement acyclicity for trees.

Existence of Nash equilibria when c is ranked worst

Boros and Gurvich (2003) showed that (in a more general model) Nash equilibria always exist when c is ranked worst by all players and either

- $n = 2$.
- There are only three outcomes.
- Every player controls only one position.

Conjecture (Boros and Gurvich (2003))

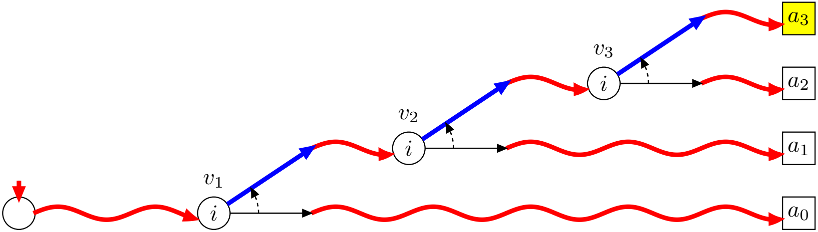
Positional games have Nash equilibria when all players rank c worst.

This conjecture would follow from there being no c-free improvement cycles.

Improvement acyclicity	Tree	DAG	$n = 2$	$n = 3$, c-free	$n = 3$	$n \geq 4$, c-free
Standard	NO†					
Restricted	YES†	NO				
Restricted **last step**	YES‡		NO			
Restricted **strong**	YES			OPEN	NO	

†Kukushkin (2002)
‡Kukushkin (2008, private communication)

We also show non-existence of **subgame perfect** equilibria.

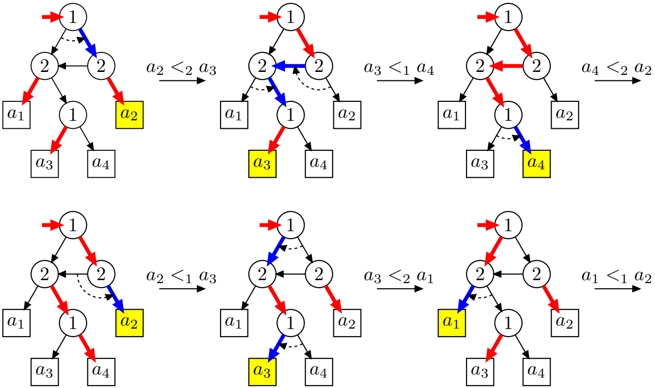


Inside play restriction (Kukushkin, 2002): Players may only change moves along the resulting **play**.

Let $\{a_0, a_1, \ldots, a_m\}$ be the sequence of intermediate outcomes. We define the following types of restricted improvements performed by player i:

Standard improvement: $a_m >_i a_0$.
Last step improvement: $a_m >_i a_{m-1}$.
Strong improvement: $a_m >_i a_j$, for all $j = 0, \ldots, m-1$.

Restricted last step improvement: The last change along the play improves the outcome.



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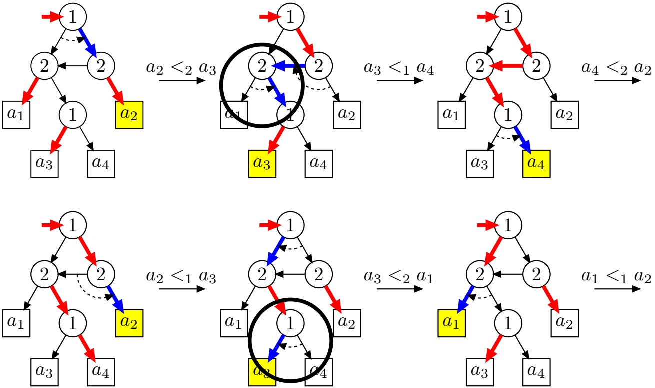

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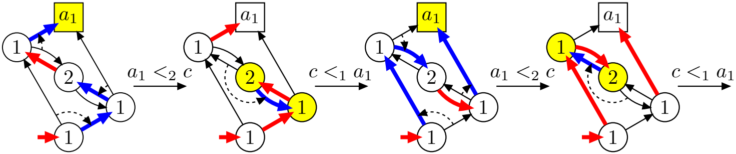
Restricted strong improvement: The final outcome is strictly better than anything seen before during changes along the play.



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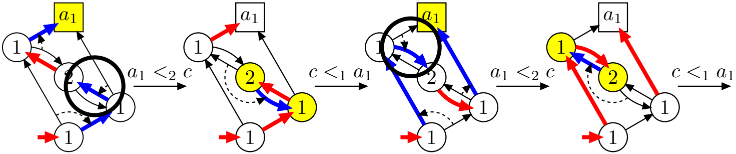

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$2: \; a_1 < c$

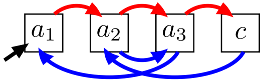
Theorem

Two-player positional games have no restricted strong improvement cycles.

Proof: By contradiction, i.e. assume we have a restricted strong improvement cycle in a *minimal* graph.

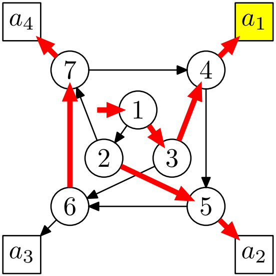
Consider the Eulerian multigraph over outcomes with edges corresponding to improvements. The subgraphs corresponding to improvements of players 1 and 2, respectively, must be acyclic.

The preferred outcome of either of the players must be terminal, and moves leading to such terminals can be fixed, contradicting minimality.



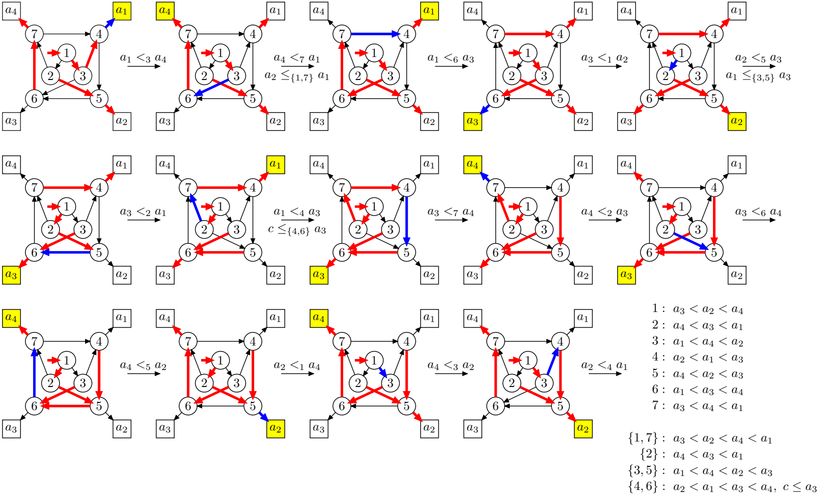
We give an example of a 14 step restricted strong improvement cycle.

- 7 players each controlling 1 vertex and having 2 choices, i.e. very few possible improvements.
- Can be brought down to 4 players by forming coalitions $\{1, 7\}$, $\{2\}$, $\{3, 5\}$ and $\{4, 6\}$.
- c-free.



We also give a 3 player example that is not c-free, but with the players having more possible improvements.

Restricted strong improvement cycle, c-free, $n = 4$



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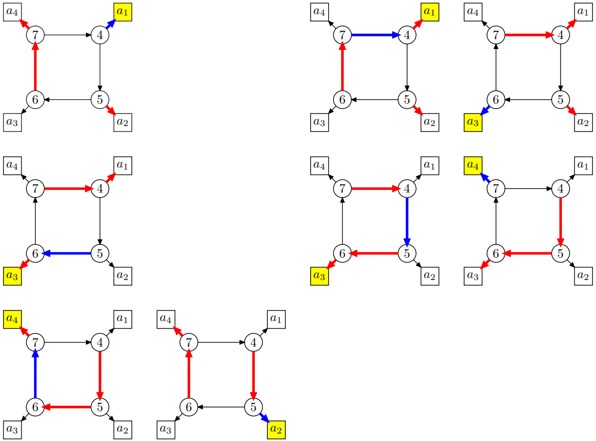

Open problems

Existence of Nash equilibria for $n \geq 3$ (even when all players rank c worst):

- **Prove.** An algorithmic approach might be successful. Algorithms would also be interesting in their own right.
- **Disprove.** Find instance with no equilibrium. It might be useful to consider the $n = 4$, c-free example.

Improvement acyclicity:

- Find a c-free restricted strong improvement cycle for $n = 3$.
- Come up with further restrictions involving the order in which players get to improve.

The end

Thank you!