Bounds on Unextendible Product Bases

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Abstract

An Unextendible Product Basis (UPB) is a set of orthogonal separable states that span a subspace of some Hilbert-space such that no separable state exists in the orthogonal complement of the subspace. UPBs were introduced by Bennett, DiVincenzo, Mor, Shor, Smolin, and Terhal [2] motivated by the relationship between UPBs and bound entanglement, and because UPBs have a new nonlocal property, namely that they are indistinguishable under local measurements and classical communication.

In [2] Bennett, DiVincenzo, Mor, Shor, Smolin, and Terhal prove the first general lower bound on the size of a UPB. In [1] Alon and Lovász give a sufficient and necessary condition for this lower bound being tight.

In my work I show the first UPB which is minimal in a Hilbert-space where the lower bound [2] is not tight.

In [3] DiVincenzo, Mor, Shor, Smolin, and Terhal introduce orthogonality graphs as a representation of the orthogonality structures of UPBs. I show that orthogonality graphs have special characteristics whenever a UPB is minimal according to the lower bound of [2].

It is an open question whether there is a simple characterisation of bound entangled states associated with minimal UPBs.

1. UPBs

Definition: Product basis

Let $\mathbb{C}^d = \otimes_{k=1}^m \mathbb{C}^{d_k}$ be an m-partite complex vector space. A *product basis* (PB) is a set S of orthogonal product states (separable states) spanning a subspace $\mathbb{C}^d_S \subseteq \mathbb{C}^d$ [2].

Definition: Unextendible Product Basis

Let S be a PB that spans a proper subspace $\mathbb{C}^d_S \subset \mathbb{C}^d$. If the orthogonal complement $\mathbb{C}^{d\perp}_S$ contains no product state, then S is said to be an *unextendible product basis* (UPB) of $\mathbb{C}^d$ [2].

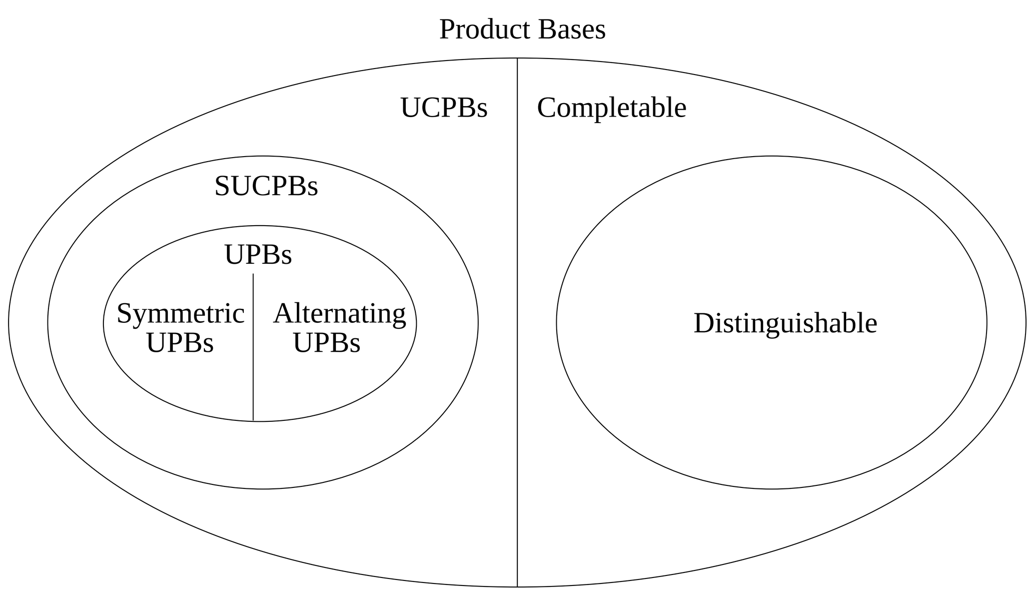

Figure 1: *Product bases fall into several classes.*

Completable product bases can be extended such that they span the full Hilbert-space. *Distinguishable product bases* are product bases where the states can be distinguished by local operations and classical communication. *Uncompletable product bases* (UCPBs), are product bases that might be extended with more orthogonal product states, but never to a full basis. *Strongly uncompletable product bases* (SUCPBs) cannot be completed even in a larger Hilbert-space. Finally *unextendible product bases* (UPBs) fall into two categories, symmetric and alternating which relates to minimality of unextendible product bases.

2. Bound Entanglement

Bound entanglement is interesting in that no singlet can be distilled from a bound entangled state. That is, given a bound entangled state no procedure exists such that a singlet can be made only by local operations and classical communication between parties holding each their part of the bound entangled state.

Theorem

Let $S = \{|\psi_1\rangle, ..., |\psi_n\rangle\}$ be a UPB of an m-partite complex vector space $\mathbb{C}^d$. Then the state corresponding to the uniform mixture on the orthogonal complement to S,

$$\bar{\rho} = \frac{1}{d-n}(\mathbb{I} - \sum_{j=1}^{n} |\psi_j\rangle\langle\psi_j|), \qquad (1)$$

has bound entanglement between the m parties of $\mathbb{C}^d$ [2].

UPBs thus give us an explicit way to construct bound entangled states.

3. Graph Representation

The orthogonality graph of a product basis is a graph with one vertex for each state of the product basis. Each part of the m-partite vector space is represented by a colour, and an edge between two vertices is coloured by the colour of the parties for which the two states are orthogonal.

Alice		Bob
$\lvert 0\rangle$	$\otimes$	$\lvert 0\rangle$,
$\lvert 1\rangle$	$\otimes$	$\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$,
$\lvert 1\rangle$	$\otimes$	$\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)$.

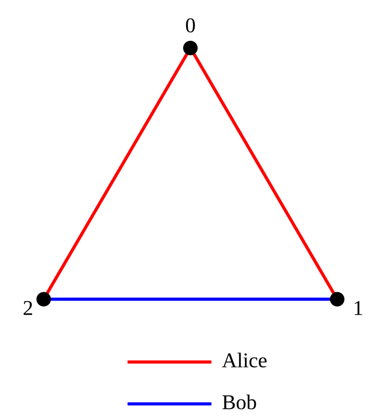

Figure 2: *A simple orthogonality graph of three orthogonal product states living in* $2 \otimes 2$.

4. Bounds on Cardinality

In [2] Bennett, DiVincenzo, Mor, Shor, Smolin, and Terhal prove the first lower bound on the size of a UPB. The lower bound implicates that all UPBs have at least as many states as denoted by Equation 2.

Theorem: Lower bound on cardinality

Let S be a UBP of an m-partite complex vector space $\mathbb{C}^d = \otimes_{k=1}^m \mathbb{C}^{d_k}$. Then S consists of at least

$$n \geq \sum_{k=1}^{m}(d_k - 1) + 1 \qquad (2)$$

product states [2].

In [1] Alon and Lovász prove a necessary and sufficient criterion for the lower bound to be tight. The criterion implies that the lower bound is tight in Hilbert-spaces in which either $\sum_{k=1}^m (d_k - 1) + 1$ is even or all d_k are odd.

Some examples of Hilbert spaces where the lower bound is not tight are $4 \otimes 4$, $4 \otimes 6$, and $2 \otimes 2 \otimes 3$.

Theorem: Alon-Lovász criterion

Let $\mathbb{C}^d = \otimes_{k=1}^m \mathbb{C}^{d_k}$ be an m-partite complex vector space, $m \geq 2$. A UPB of $\mathbb{C}^d$ of cardinality $\sum_{k=1}^m (d_k - 1) + 1$ exists if and only if *none* of the following is true

- $m = 2$ and $2 \in \{d_1, d_2\}$.
- $\sum_{k=1}^m (d_k - 1) + 1$ is odd and at least one of d_k is even.

[1].

In my work with unextendible product bases I have found an unextendible product basis with 8 states in $4 \otimes 4$, which is the first known unextendible product basis that is minimal in a vector space in which the lower bound is not tight.

The **Min4x4** unextendible product basis lives in $\mathbb{R}^4 \otimes \mathbb{R}^4$.

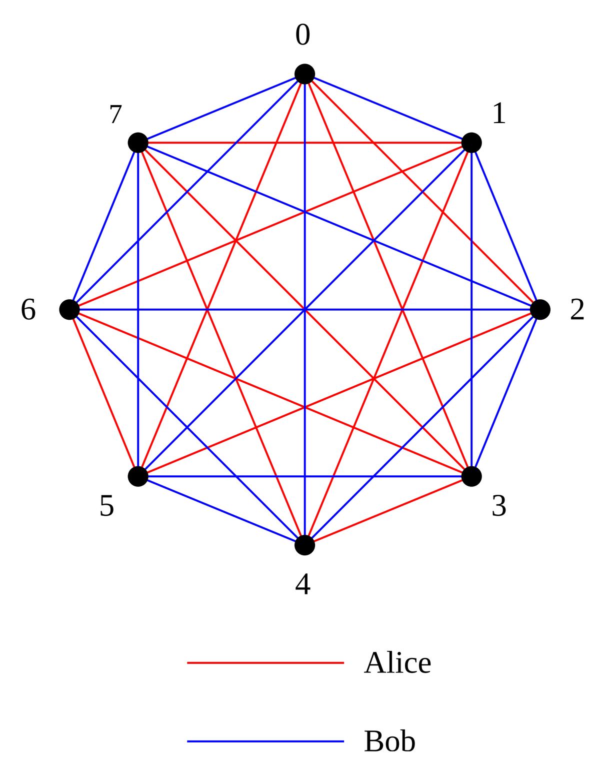

Figure 3: *The graph representation of the orthogonality of the* **Min4x4** *UPB.*

In the following all states are to be multiplied by a normalisation factor.

$$
\begin{aligned}
&|0\rangle - 3|1\rangle + |2\rangle + |3\rangle && \otimes\ |1\rangle - (3+\sqrt{2})|2\rangle - (1+\sqrt{2})|3\rangle,\\
&|0\rangle && \otimes\ |0\rangle,\\
&|1\rangle + 2|2\rangle + |3\rangle && \otimes\ |0\rangle + (\sqrt{2}-1)|2\rangle + |3\rangle,\\
&|0\rangle - |3\rangle && \otimes\ |1\rangle,\\
&|1\rangle && \otimes\ -|0\rangle + (1+\sqrt{2})|1\rangle + |3\rangle,\\
&3|0\rangle + |1\rangle - |2\rangle + |3\rangle && \otimes\ |2\rangle,\\
&|1\rangle + |2\rangle && \otimes\ |0\rangle + |1\rangle + |2\rangle - \sqrt{2}|3\rangle,\\
&|2\rangle && \otimes\ -|0\rangle + (1+\sqrt{2})|1\rangle + |3\rangle.
\end{aligned}
$$

Notice that the four brown states on Alice's side do not span the local system of Alice. But the four remaining states span the local system of Bob. In order for an unextendible product basis to conform to the lower bound of [2] any d_k states of the kth party have to span $\mathbb{C}^{d_k}$. The **Min4x4** UPB does not have this property.

Because of the relationship to bound entanglement an upper bound on the cardinality can be given.

Lemma: No rank two bound entangled state

All bipartite bound entangled states have rank greater than two [4].

Theorem

Let S be a UPB of an m-partite Hilbert space $\mathbb{C}^d = \otimes_{k=1}^m \mathbb{C}^{d_k}$. Then S consists of at most

$$n \leq \left(\prod_{k=1}^{m} k\right) - 3 \qquad (3)$$

product states [4].

5. Alternating and Symmetric UPBs

In my work I have found an alternative characterisation of the UPBs that satisfies the lower bound from [2]. This characterisation have a nice representation in the orthogonality graphs.

Definition: Symmetric UPB

Let $\mathbb{C}^d = \otimes_{k=1}^m \mathbb{C}^{d_k}$ be an m-partite complex vector space. A UPB S of $\mathbb{C}^d$ is a *symmetric UPB* if and only if for all $k = 1, \ldots, m$ and all subsets S' of S of cardinality d_k the states of the kth party of S' span $\mathbb{C}^{d_k}$.

All UPBs that are not symmetric are alternating.

Definition: Alternating UPB

Let $\mathbb{C}^d = \otimes_{k=1}^m \mathbb{C}^{d_k}$ be an m-partite complex vector space. A UPB S of $\mathbb{C}^d$ is an *alternating UPB* if and only if for some disjoint subsets $S_1 \cup \ldots \cup S_m = S$ of S at least one subset S_k has d_k or more states but does not span $\mathbb{C}^{d_k}$.

Figure 4 (A) shows the orthogonality graph of a symmetric UPB in $4 \otimes 5$. Notice that all vertices have the same out degree in each party (4 for Alice and 3 for Bob). Figure 4 (B) shows the orthogonality graph of the **Min4x4** UPB, which is an alternating UPB. Notice that not all the vertices have the same degree.

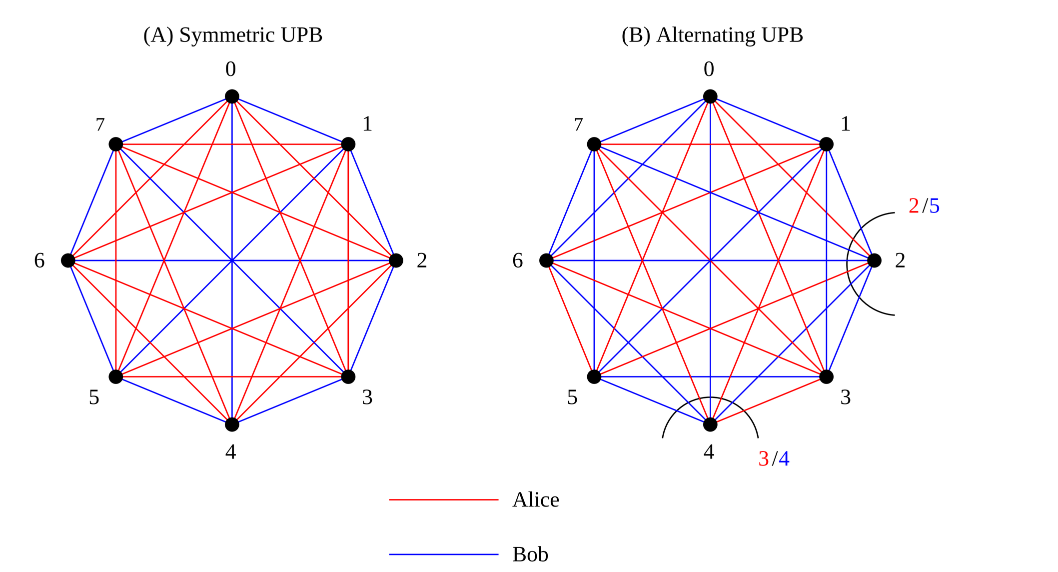

Figure 4: *(A) Symmetric UPB in* $4 \otimes 5$. *All vertices in a symmetric UPB have the same degree. (B) Alternating UPB in* $4 \otimes 4$. *The vertices of an alternating UPB does not have the same degree.*

Theorem

Let S be a UPB of an m-partite complex vector space $\mathbb{C}^d = \otimes_{k=1}^m \mathbb{C}^{d_k}$. Then S is symmetric if and only if S has cardinality $n = \sum_{k=1}^m (d_k - 1) + 1$, which is minimal.

References

[1] Alon and Lovász. Unextendible product bases. *J. Combinatorial Theory, Ser. A*, 95:169–179, 2001.

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