Strong Correspondence for HOPLA

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Abstract

We show that the operational semantics for HOPLA is in strong correspondence with its presheaf semantics [2, 1]. The proof is a fairly standard logical relations proof, exploiting the path semantics of the language [3, 4].

Strong correspondence can be proved for full HOPLA by making use of the path semantics to get a logical relations proof off the ground. We'll use the notation $[\![-]\!]$ for the path semantics and $[\![-]\!]^{\mathbf{Set}}$ for the presheaf semantics. The proof uses logical predicates $A_{\mathbb{P}}(p,t)$, where $p : \mathbb{P}$ is a formal path of the path semantics, and t is a closed term of type $\mathbb{P}$. By structural induction on paths, we define:

$$\begin{aligned}
A_{\mathbb{P}\to\mathbb{Q}}(P \mapsto q, t) &\iff_{\text{def}} \forall u.\ (A_{\mathbb{P}}(P,u) \implies A_{\mathbb{Q}}(q, t\ u)) \\
A_{\Sigma_{\alpha\in A}\mathbb{P}_\alpha}(\beta p, t) &\iff_{\text{def}} A_{\mathbb{P}_\beta}(p, \pi_\beta t) \\
A_{!\mathbb{P}}(P,t) &\iff_{\text{def}} \begin{cases} ([\![t]\!]^{\mathbf{Set}} \cong \Sigma_d [\![!t_d]\!]^{\mathbf{Set}}) \text{ and} \\ (!\mathbb{P} : t \xrightarrow{!} t' : \mathbb{P} \implies A_{\mathbb{P}}(P,t')) \end{cases} \\
A_{\mu_j \vec{T}.\vec{\mathbb{T}}}(abs\, p, t) &\iff_{\text{def}} A_{\mathbb{T}_j[\mu\vec{T}.\vec{\mathbb{T}}/\vec{T}]}(p, rep\, t)
\end{aligned} \tag{1}$$

Here, the sum index d ranges over derivations of $!\mathbb{P} : t \xrightarrow{!} t_d : \mathbb{P}$ and the logical predicates are extended to sets of paths $X \subseteq \mathbb{P}$ by

$$A_{\mathbb{P}}(X,t) \iff_{\text{def}} \forall p \in X.\ A_{\mathbb{P}}(p,t)\ . \tag{2}$$

It will be convenient to extend the logical predicates to actions:

$$\frac{A_{\mathbb{P}}(P,u) \quad A(q, \mathbb{Q} : a : \mathbb{P}', P')}{A(P \mapsto q, \mathbb{P} \to \mathbb{Q} : u \mapsto a : \mathbb{P}', P')} \quad \frac{A(p, \mathbb{P}_\beta : a : \mathbb{P}', P') \quad \beta \in A}{A(\beta p, \Sigma_{\alpha\in A}\mathbb{P}_\alpha : \beta a : \mathbb{P}', P')}$$
$$\frac{}{A(P, !\mathbb{P} : ! : \mathbb{P}, P)} \quad \frac{A(p, \mathbb{T}_j[\mu\vec{T}.\vec{\mathbb{T}}/\vec{T}] : a : \mathbb{P}', P')}{A(abs\, p, \mu_j\vec{T}.\vec{\mathbb{T}} : abs\, a : \mathbb{P}', P')} \tag{3}$$

In a judgement $A(p, \mathbb{P} : a : \mathbb{P}', P')$ we have that $\mathbb{P} : a : \mathbb{P}'$ is an action while $p : \mathbb{P}$ and $P' : !\mathbb{P}'$ are paths, the latter uniquely determined by p and a; it can be understood intuitively as representing the part of p which has not yet been "realised" by the action a. The following lemma is what makes these judgements useful:

Lemma 0.1 Suppose $\vdash t : \mathbb{P}$ and $p : \mathbb{P}$. Then $A_{\mathbb{P}}(p, t)$ iff for all actions $\mathbb{P} : a : \mathbb{P}'$ with $A(p, \mathbb{P} : a : \mathbb{P}', P')$ we have $A_{!\mathbb{P}'}(P', a^* t)$.

Proof. By structural induction on p.

Function space. Suppose $\vdash t : \mathbb{P} \to \mathbb{Q}$ and $P \mapsto q : \mathbb{P} \to \mathbb{Q}$. Assume $A_{\mathbb{P}\to\mathbb{Q}}(P \mapsto q, t)$ and let $\mathbb{P} \to \mathbb{Q} : u \mapsto a : \mathbb{P}'$ be any action with $A(P \mapsto q, \mathbb{P} \to \mathbb{Q} : u \mapsto a : \mathbb{P}', P')$. We then have $A_{\mathbb{P}}(P, u)$ and hence $A_{\mathbb{Q}}(q, t\ u)$ by definition of $A_{\mathbb{P}\to\mathbb{Q}}$. We also have $A(q, \mathbb{Q} : a : \mathbb{P}', P')$ and so $A_{!\mathbb{P}'}(P', a^*(t\ u))$, the same as $A_{!\mathbb{P}'}(P', (u \mapsto a)^* t)$, by the induction hypothesis for q.

Conversely, suppose that $A_{!\mathbb{P}'}(P', (u \mapsto a)^* t)$ for any action $\mathbb{P} \to \mathbb{Q} : u \mapsto a : \mathbb{P}'$ satisfying $A(P \mapsto q, \mathbb{P} \to \mathbb{Q} : u \mapsto a : \mathbb{P}', P')$. To show $A_{\mathbb{P}\to\mathbb{Q}}(P \mapsto q, t)$, let $\vdash u : \mathbb{P}$ with $A_{\mathbb{P}}(P, u)$ be given. We need to show that $A_{\mathbb{Q}}(q, t\ u)$ and for that, we'll invoke the induction hypothesis for q. So let $\mathbb{Q} : a : \mathbb{P}'$ with $A(q, \mathbb{Q} : a : \mathbb{P}', P')$ be given. We then deduce $A(P \mapsto q, \mathbb{P} \to \mathbb{Q} : u \mapsto a : \mathbb{P}', P')$. By the assumption, $A_{!\mathbb{P}'}(P', (u \mapsto a)^* t)$ which is the same as saying $A_{!\mathbb{P}'}(P', a^*(t\ u))$. The induction hypothesis now applies and yields $A_{\mathbb{Q}}(q, t\ u)$ as wanted.

Sum type. Suppose $\vdash t : \Sigma_{\alpha\in A}\mathbb{P}_\alpha$ and $\beta p : \Sigma_{\alpha\in A}\mathbb{P}_\alpha$. Assume $A_{\Sigma_{\alpha\in A}\mathbb{P}_\alpha}(\beta p, t)$ and let $\Sigma_{\alpha\in A}\mathbb{P}_\alpha : \beta a : \mathbb{P}'$ be any action with $A(\beta p, \Sigma_{\alpha\in A}\mathbb{P}_\alpha : \beta a : \mathbb{P}', P')$. We then have both $A_{\mathbb{P}_\beta}(p, \pi_\beta t)$ and $A(p, \mathbb{P}_\beta : a : \mathbb{P}', P')$ and so $A_{!\mathbb{P}'}(P', a^*(\beta t))$, the same as $A_{!\mathbb{P}'}(P', (\beta a)^* t)$, by the induction hypothesis for p.

Conversely, suppose that $A_{!\mathbb{P}'}(P', (\beta a)^* t)$ for any action $\Sigma_{\alpha\in A}\mathbb{P}_\alpha : \beta a : \mathbb{P}'$ satisfying $A(\beta p, \Sigma_{\alpha\in A}\mathbb{P}_\alpha : \beta a : \mathbb{P}', P')$. We need to show $A_{\mathbb{P}_\beta}(p, \pi_\beta t)$ and for that, we'll invoke the induction hypothesis for p. So let $\mathbb{P}_\beta : a : \mathbb{P}'$ with $A(p, \mathbb{P}_\beta : a : \mathbb{P}', P')$ be given. We then deduce $A(\beta p, \Sigma_{\alpha\in A}\mathbb{P}_\alpha : \beta a : \mathbb{P}', P')$. By the assumption, $A_{!\mathbb{P}'}(P', (\beta a)^* t)$ which is the same as saying $A_{!\mathbb{P}'}(P', a^*(\pi_\beta t))$. The induction hypothesis now applies and yields $A_{\mathbb{P}_\beta}(p, \pi_\beta t)$ as wanted.

Prefix type. Immediate, as there is only one judgment of the form $A(P, !\mathbb{P} : a : \mathbb{P}', P')$, namely $A(P, !\mathbb{P} : ! : \mathbb{P}, P)$, and $!^* t \equiv t$.

Recursive types. Similar to sum type.

□

To prove the Main Lemma below, we'll need some facts about the operational semantics and the presheaf denotational semantics:

Lemma 0.2 $\mathbb{P} : t \xrightarrow{a} t' : \mathbb{P}' \iff !\mathbb{P}' : a^* t \xrightarrow{!} t' : \mathbb{P}'$ with a bijective correspondence between derivations.

Proposition 0.3 In the presheaf semantics we have the following isomorphisms:

$$[\![rec\,x.t]\!]^{\mathbf{Set}} \cong [\![t[rec\,x.t/x]]\!]^{\mathbf{Set}} \tag{4}$$
$$[\![(\lambda x.t)\ u]\!]^{\mathbf{Set}} \cong [\![t[u/x]]\!]^{\mathbf{Set}} \tag{5}$$
$$[\![\pi_\beta(\beta t)]\!]^{\mathbf{Set}} \cong [\![t]\!]^{\mathbf{Set}} \tag{6}$$
$$[\![[!u > !x \Rightarrow t]]\!]^{\mathbf{Set}} \cong [\![t[u/x]]\!]^{\mathbf{Set}} \tag{7}$$

We can now prove:

Lemma 0.4 (Main Lemma) For all terms $\vdash t : \mathbb{P}$ we have $A_{\mathbb{P}}([\![t]\!], t)$.

Proof. By structural induction on terms using the induction hypothesis

> Suppose $\Gamma \vdash t : \mathbb{P}$ and let $\vdash s_j : \mathbb{P}_j$ and $X_j \subseteq \mathbb{P}_j$ with $A_{\mathbb{P}_j}(X_j, s_j)$ for $1 \le j \le k$. Then $A_{\mathbb{P}}([\![t]\!]X, t[s])$.

Here, as well as below, we use the shorthands Γ for $x_1 : \mathbb{P}_1, \ldots, x_k : \mathbb{P}_k$, X for $(X_1, \ldots, X_k)$, and $[s]$ for $[s_1/x_1, \ldots, s_k/x_k]$.

By using Lemmas 0.1 and 0.2, we may reformulate the induction hypothesis into an equivalent form:

> Suppose $\Gamma \vdash t : \mathbb{P}$ and let $\vdash s_j : \mathbb{P}_j$ and $X_j \subseteq \mathbb{P}_j$ with $A_{\mathbb{P}_j}(X_j, s_j)$ for $1 \le j \le k$. Then for any $p \in [\![t]\!]X$ and any action a with $A(p, \mathbb{P} : a : \mathbb{P}', P')$, we have $A_{!\mathbb{P}'}(P', a^* t[s])$, i.e.
>
> $$[\![a^* t[s]]\!]^{\mathbf{Set}} \cong \Sigma_d [\![!t_d]\!]^{\mathbf{Set}} \text{ and} \tag{8}$$
> $$\mathbb{P} : t[s] \xrightarrow{a} t' : \mathbb{P}' \implies A_{\mathbb{P}'}(P', t') \tag{9}$$
>
> —where d ranges over derivations $!\mathbb{P}' : t[s] \xrightarrow{a} t_d : \mathbb{P}'$.

We'll silently use whichever form is more convenient for each case below:

Variable. Immediate.

Recursive definition. Let $\Gamma \vdash rec\,x.t : \mathbb{P}$ and $\vdash s_j : \mathbb{P}_j$ with $A_{\mathbb{P}}(X_j, s_j)$ for $1 \le j \le k$. We must show that $A_{\mathbb{P}}([\![rec\,x.t]\!]X, rec\,x.t[s])$. Now, $[\![rec\,x.t]\!]X = (fix\ f)X$ where f maps $g : \Gamma \to \mathbb{P}$ to the composition

$$\Gamma \xrightarrow{\Delta_\Gamma} \Gamma \,\&\, \Gamma \xrightarrow{1_\Gamma \& g} \Gamma \,\&\, \mathbb{P} \xrightarrow{t} \mathbb{P}\ . \tag{10}$$

We'll show by induction on n that $A_{\mathbb{P}}((f^n \varnothing)X, rec\,x.t[s])$ for all $n \in \omega$. Having done so we may argue as follows: Since

$$[\![rec\,x.t]\!]X = (fix\ f)X = (\textstyle\bigcup_{n\in\omega} f^n\varnothing)X = \bigcup_{n\in\omega}((f^n\varnothing)X)\ , \tag{11}$$

we have that $p \in [\![rec\,x.t]\!]X$ implies the existence of an $n \in \omega$ such that $p \in (f^n\varnothing)X$. Therefore $A_{\mathbb{P}}([\![rec\,x.t]\!]X, rec\,x.t[s])$ as wanted.

Basis. Here, $(f^0\varnothing)X = \varnothing$. By the definition of $A_{\mathbb{P}}$ extended to sets of paths, we get $A_{\mathbb{P}}(\varnothing, t)$ for any type $\mathbb{P}$ and term $\vdash t : \mathbb{P}$.

Step. We must show that $A_{\mathbb{P}}((f^{n+1}\varnothing)X, rec\, x.t[s])$. By the denotational semantics, $(f^{n+1}\varnothing)X = [\![t]\!](X, (f^n\varnothing)X)$ and by the induction hypothesis for the mathematical induction, $A_{\mathbb{P}}((f^n\varnothing)X, rec\, x.t[s])$ such that the induction hypothesis for the structural induction applies to the term t with $(X, (f^n\varnothing)X)$ and the substitution $[s][rec\, x.t[s]/x]$.

Let $p \in (f^{n+1}\varnothing)X$ and let a be any action such that $A(p, \mathbb{P} : a : \mathbb{P}', P')$. With d ranging over derivations $\mathbb{P} : rec\, x.t[s] \xrightarrow{a} t_d : \mathbb{P}'$ and d' over derivations $\mathbb{P} : t[s][rec\, x.t[s]/x] \xrightarrow{a} t_{d'} : \mathbb{P}'$, we have $\Sigma_d[\![!t_d]\!]^{\mathbf{Set}} \cong \Sigma_{d'}[\![!t_{d'}]\!]^{\mathbf{Set}}$ by the operational rules, and hence:

$$\begin{array}{ll} [\![a^*(rec\, x.t[s])]\!]^{\mathbf{Set}} & \\ \cong [\![a^*t[s][rec\, x.t[s]/x]]\!]^{\mathbf{Set}} & \text{(by (4))} \\ \cong \Sigma_{d'}[\![!t_{d'}]\!]^{\mathbf{Set}} & \text{(ind. hyp.)} \\ \cong \Sigma_d[\![!t_d]\!]^{\mathbf{Set}} & \end{array} \tag{12}$$

Further, if $\mathbb{P} : rec\, x.t[s] \xrightarrow{a} t' : \mathbb{P}'$, then we have $\mathbb{P} : t[s][rec\, x.t[s]/x] \xrightarrow{a} t' : \mathbb{P}'$ and so by the induction hypothesis for the structural induction, $A_{\mathbb{P}'}(P', t')$ as wanted.

We conclude $A_{\mathbb{P}}((f^{n+1}\varnothing)X, rec\, x.t[s])$ and the mathematical induction is complete.

Nondeterministic sum. Consider $\Gamma \vdash \Sigma_{i\in I}t_i : \mathbb{P}$. Let $X_j \subseteq \mathbb{P}_j$ and $\vdash s_j : \mathbb{P}_j$ with $A_{\mathbb{P}_j}(X_j, s_j)$ for $1 \leq j \leq k$. Suppose $p \in [\![\Sigma_{i\in I}t_i]\!]X$ and let a be any action satisfying $A(p, \mathbb{P} : a\mathbb{P}', P')$. Letting d range over derivations $\mathbb{P} : \Sigma_{i\in I}t_i[s] \xrightarrow{a} t_d : \mathbb{P}'$ and d_i over derivations $\mathbb{P} : t_i[s] \xrightarrow{a} t_{d_i} : \mathbb{P}'$ for each $i \in I$, we have $\Sigma_d[\![!t_d]\!]^{\mathbf{Set}} \cong \Sigma_{i\in I}\Sigma_{d_i}[\![!t_{d_i}]\!]^{\mathbf{Set}}$ by the operational rules, and hence:

$$\begin{array}{ll} [\![a^*\Sigma_{i\in I}t_i[s]]\!]^{\mathbf{Set}} & \\ \cong a^*\Sigma_{i\in I}[\![t_i[s]]\!]^{\mathbf{Set}} & \\ \cong \Sigma_{i\in I}[\![a^*t_i[s]]\!]^{\mathbf{Set}} & \text{(linearity of } a^*) \\ \cong \Sigma_{i\in I}\Sigma_{d_i}[\![!t_{d_i}]\!]^{\mathbf{Set}} & \text{(ind. hyp.)} \\ \cong \Sigma_d[\![!t_d]\!]^{\mathbf{Set}} & \end{array} \tag{13}$$

Further, if $\mathbb{P} : \Sigma_{i\in I}t_i[s] \xrightarrow{a} t' : \mathbb{P}'$, then for some $j \in I$ we have $\mathbb{P} : t_j[s] \xrightarrow{a} t' : \mathbb{P}'$ and so by the induction hypothesis, $A_{\mathbb{P}'}(P', t')$ as wanted.

Abstraction. Consider $\Gamma \vdash \lambda x.t : \mathbb{P} \to \mathbb{Q}$. Let $X_j \subseteq \mathbb{P}_j$ and $\vdash s_j : \mathbb{P}_j$ with $A_{\mathbb{P}_j}(X_j, s_j)$ for $1 \leq j \leq k$. Suppose $P \mapsto q \in [\![\lambda x.t]\!]X$ and let $u \mapsto a$ be any action satisfying $A(P \mapsto q, \mathbb{P} \to \mathbb{Q} : u \mapsto a : \mathbb{P}', P')$. Since this implies that $A_{\mathbb{P}}(P, u)$, the induction hypothesis applies to the term t with (X, P) and substitution $[s][u/x]$. Letting d range over derivations $\mathbb{P} \to \mathbb{Q} : \lambda x.t[s] \xrightarrow{u \mapsto a} t_d : \mathbb{P}'$ and d' over derivations $\mathbb{P} : t[s][u/x] \xrightarrow{a} t_{d'} : \mathbb{P}'$, we have

$\Sigma_d[\![!t_d]\!]^{\mathbf{Set}} \cong \Sigma_{d'}[\![!t_{d'}]\!]^{\mathbf{Set}}$ by the operational rules, and hence:

$$\begin{array}{ll} [\![(u \mapsto a)^*(\lambda x.t[s])]\!]^{\mathbf{Set}} & \\ \cong [\![a^*(\lambda x.t[s]\ u)]\!]^{\mathbf{Set}} & \text{(def. of } (u \mapsto a)^*) \\ \cong [\![a^*(t[s][u/x])]\!]^{\mathbf{Set}} & \text{(by (5))} \\ \cong \Sigma_{d'}[\![!t_{d'}]\!]^{\mathbf{Set}} & \text{(ind. hyp.)} \\ \cong \Sigma_d[\![!t_d]\!]^{\mathbf{Set}} & \end{array} \tag{14}$$

Further, if $\mathbb{P} \to \mathbb{Q} : \lambda x.t[s] \xrightarrow{u \mapsto a} t' : \mathbb{P}'$, then $\mathbb{Q} : t[s][u/x] \xrightarrow{a} t' : \mathbb{P}'$ and so by the induction hypothesis, $A_{\mathbb{P}'}(P', t')$ as wanted.

Application. Consider $\Gamma \vdash t\ u : \mathbb{Q}$. Let $X_j \subseteq \mathbb{P}_j$ and $\vdash s_j : \mathbb{P}_j$ with $A_{\mathbb{P}_j}(X_j, s_j)$ for $1 \leq j \leq k$. Suppose $q \in [\![t\ u]\!]X$. Then for some $P : !\mathbb{P}$, we have $P \mapsto q \in [\![t]\!]X$ and $P \subseteq [\![u]\!]X$. By the induction hypothesis, $A_{\mathbb{P} \to \mathbb{Q}}(P \mapsto q, t[s])$ and $A_{\mathbb{P}}(P, u[s])$. The definition of $A_{\mathbb{P} \to \mathbb{Q}}$ yields $A_{\mathbb{Q}}(q, t[s]\ u[s])$ as wanted.

Injection. Consider $\Gamma \vdash \beta t : \Sigma_{\alpha \in A}\mathbb{P}_\alpha$. Let $X_j \subseteq \mathbb{P}_j$ and $\vdash s_j : \mathbb{P}_j$ with $A_{\mathbb{P}_j}(X_j, s_j)$ for $1 \leq j \leq k$. Suppose $\beta p \in [\![\beta t]\!]X$ and let βa be any action such that $A(\beta p, \Sigma_{\alpha \in A}\mathbb{P}_\alpha : \beta a : \mathbb{P}', P')$. Letting d range over derivations $\Sigma_{\alpha \in A}\mathbb{P}_\alpha : \beta t[s] \xrightarrow{\beta a} t_d : \mathbb{P}'$ and d' over derivations $\mathbb{P}_\beta : t \xrightarrow{a} t_{d'} : \mathbb{P}'$, we have $\Sigma_d[\![!t_d]\!]^{\mathbf{Set}} \cong \Sigma_{d'}[\![!t_{d'}]\!]^{\mathbf{Set}}$ by the operational rules, and hence:

$$\begin{array}{ll} [\![(\beta a)^*(\beta t[s])]\!]^{\mathbf{Set}} & \\ \cong [\![a^*(\pi_\beta(\beta t[s]))]\!]^{\mathbf{Set}} & \text{(def. of } (\beta a)^*) \\ \cong [\![a^*(t[s])]\!]^{\mathbf{Set}} & \text{(by (6))} \\ \cong \Sigma_{d'}[\![!t_{d'}]\!]^{\mathbf{Set}} & \text{(ind. hyp.)} \\ \cong \Sigma_d[\![!t_d]\!]^{\mathbf{Set}} & \end{array} \tag{15}$$

Further, if $\Sigma_{\alpha \in A}\mathbb{P}_\alpha : \beta t[s] \xrightarrow{\beta a} t' : \mathbb{P}'$, then $\mathbb{P}_\beta : t[s] \xrightarrow{a} t' : \mathbb{P}'$ and so by the induction hypothesis, $A_{\mathbb{P}'}(P', t')$ as wanted.

Projection. Consider $\Gamma \vdash \pi_\beta t : \mathbb{P}_\beta$. Let $X_j \subseteq \mathbb{P}_j$ and $\vdash s_j : \mathbb{P}_j$ with $A_{\mathbb{P}_j}(X_j, s_j)$ for $1 \leq j \leq k$. Suppose $p \in [\![\pi_\beta t]\!]X$ so that $\beta p \in [\![t]\!]X$. By the induction hypothesis, $A_{\Sigma_{\alpha \in A}\mathbb{P}_\alpha}(\beta p, t[s])$ and so by definition, $A_{\mathbb{P}_\beta}(p, \pi_\beta t[s])$ as wanted.

Prefixing. Consider $\Gamma \vdash !t : !\mathbb{P}$. Let $X_j \subseteq \mathbb{P}_j$ and $\vdash s_j : \mathbb{P}_j$ with $A_{\mathbb{P}_j}(X_j, s_j)$ for $1 \leq j \leq k$. Suppose $P \in [\![!t]\!]X$ so that $P \subseteq [\![t]\!]X$. We must show that $A_{!\mathbb{P}}(P, !t)$. There is just one derivation $!\mathbb{P} : !t[s] \xrightarrow{!} t[s] : \mathbb{P}$ and we have the identity $[\![!^*(!t[s])]\!]^{\mathbf{Set}} = [\![!t[s]]\!]^{\mathbf{Set}}$ by definition. Further, $A_{\mathbb{P}}(P, t[s])$ follows from the induction hypothesis.

Prefix match Consider $\Gamma \vdash [u > !x \Rightarrow t] : \mathbb{Q}$. Let $X_j \subseteq \mathbb{P}_j$ and $\vdash s_j : \mathbb{P}_j$ with $A_{\mathbb{P}_j}(X_j, s_j)$ for $1 \leq j \leq k$. By renaming x if necessary, we may without loss of generality assume that x is distinct from each x_j. Suppose $q \in [\![[u > !x \Rightarrow t]]\!]X$ and let a be any action such that $A(q, \mathbb{Q} : a : \mathbb{P}', P')$. Since $q \in$

$[\![[u > !x \Rightarrow t]]\!] X$ there exists $P : !\mathbb{P}$ such that $P \subseteq [\![u]\!] X$ and $q \in [\![t]\!](X, P)$. The induction hypothesis for u yields $[\![!^* u[s]]\!]^{\mathbf{Set}} = [\![u[s]]\!]^{\mathbf{Set}} \cong \Sigma_{d'} [\![!u'_{d'}]\!]^{\mathbf{Set}}$ and $A_{\mathbb{P}}(P, u'_{d'})$ for derivations d' of $!\mathbb{P} : u[s] \xrightarrow{!} u'_{d'} : \mathbb{P}$. Letting d range over derivations $\mathbb{Q} : [u[s] > !x \Rightarrow t[s]] \xrightarrow{a} t_d : \mathbb{P}'$ and $d_{d'}$ over derivations $\mathbb{Q} : t[s][u'_{d'}/x] \xrightarrow{a} t_{d_{d'}} : \mathbb{P}'$, we have $\Sigma_d [\![!t_d]\!]^{\mathbf{Set}} \cong \Sigma_{d'} \Sigma_{d_{d'}} [\![!t_{d_{d'}}]\!]^{\mathbf{Set}}$ by the operational rules and hence,

$$
\begin{array}{lll}
[\![a^*[u[s] > !x \Rightarrow t[s]]]\!]^{\mathbf{Set}} & & \\
\cong [\![a^*[\Sigma_{d'} !u'_{d'} > !x \Rightarrow t[s]]]\!]^{\mathbf{Set}} & \text{(compositionality)} & \\
\cong a^* \Sigma_{d'} [\![t[s][u'_{d'}/x]]\!]^{\mathbf{Set}} & \text{(by (7))} & \\
\cong \Sigma_{d'} [\![a^*(t[s][u'_{d'}/x])]\!]^{\mathbf{Set}} & \text{(linearity of } a^*) & \quad (16) \\
\cong \Sigma_{d'} \Sigma_{d_{d'}} [\![!t_{d_{d'}}]\!]^{\mathbf{Set}} & \text{(ind. hyp. for } t) & \\
\cong \Sigma_d [\![!t_d]\!]^{\mathbf{Set}} & &
\end{array}
$$

Further, if $\mathbb{Q} : [u[s] > !x \Rightarrow t[s]] \xrightarrow{a} t' : \mathbb{P}'$, then $!\mathbb{P} : u[s] \xrightarrow{!} u' : \mathbb{P}$ for some u' such that $A_{\mathbb{P}}(P, u')$ by the induction hypothesis for u, and then $\mathbb{Q} : t[s][u'/x] \xrightarrow{a} t' : \mathbb{P}'$ with $A_{\mathbb{Q}}(P', t')$ by the induction hypothesis for t, as wanted.

Folding and unfolding. Similar to injection and projection.

The structural induction is complete. □

Theorem 0.5 (Strong Correspondence) Suppose $\vdash t : \mathbb{P}$ and $\mathbb{P} : a : \mathbb{P}'$. Then $a^* [\![t]\!]^{\mathbf{Set}} \cong \Sigma_d [\![!t_d]\!]^{\mathbf{Set}}$ where d ranges over derivations of $\mathbb{P} : t \xrightarrow{a} t_d : \mathbb{P}'$.

Proof. If $a^* [\![t]\!]^{\mathbf{Set}}$ is the empty presheaf, we have no derivations $\mathbb{P} : t \xrightarrow{a} t_d : \mathbb{P}'$ by soundness, as wanted.

Otherwise, the path set $a^* [\![t]\!] = [\![a^* t]\!]$ is non-empty as well, say $P' \in [\![a^* t]\!]$. By the main lemma we have $A_{!\mathbb{P}'}(P', a^* t)$ and hence, $[\![a^* t]\!]^{\mathbf{Set}} \cong \Sigma_{d'} [\![!t_{d'}]\!]^{\mathbf{Set}}$ where d' ranges over derivations of $!\mathbb{P}' : a^* t \xrightarrow{!} t_{d'} : \mathbb{P}'$. The result then follows by Lemma 0.2. □

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