

Nash Stability in Additively Separable Hedonic Games is NP-hard

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Abstract. Ballester has shown that the problem of deciding whether a Nash stable partition exists in a hedonic game with arbitrary preferences is NP-complete. In this paper we will prove that the problem remains NP-complete even when restricting to additively separable hedonic games. Bogomolnaia and Jackson have shown that a Nash stable partition exists in every additively separable hedonic game with *symmetric* preferences. We show that *computing* Nash stable partitions is hard in games with symmetric preferences. To be more specific we show that the problem of deciding whether a *non trivial* Nash stable partition exists in an additively separable hedonic game with *non-negative* and *symmetric* preferences is NP-complete. The corresponding problem concerning individual stability is also NP-complete since individually stable partitions are Nash stable and vice versa in such games.

Key words: Nash Stability, Hedonic Games, NP-Completeness

1 Introduction

In a *Coalition Formation Game* a set of players splits up in coalitions so that each player belongs to exactly one coalition. Each player prefers certain partitions¹ of the players to other partitions. If all players are satisfied with the partition in some formalized sense - or not able to move - the partition is said to be *stable*. A stable partition is called an *equilibrium*. For an overview of the field of *Coalition Formation Games* we refer to the report [7] by Hajdukova.

A given notion of stability can have limitations in terms of computability. For some types of games it might be impossible to effectively compute equilibriums on a computing device under the assumption $NP \neq P$. If a real world system is modeled using Coalition Formation Games and equilibriums with such limitations you should not expect to be able to calculate the equilibriums using a computer if the model is large. It is also an interesting question whether a real

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¹ A partition of a set N is a collection of non empty disjoint subsets of N with union N .

system is able to find an equilibrium if a computer can not find it effectively. This is the motivation for analyzing the computational complexity for a given notion of stability as also pointed out by Daskalakis and Papadimitriou in [4] and Chen and Rudra in [3]. This paper deals with proving limitations for the notion of *Nash stability* in *Additively Separable Hedonic Games*.

1.1 Additively Separable Hedonic Games

In a *hedonic game* we are given a set $N = \{1, 2, \dots, |N|\}$ of *players*. Each player $i \in N$ has a reflexive, complete and transitive *preference relation* \preceq_i on the set $N_i = \{S \subseteq N : i \in S\}$. In this way we know which *coalitions* player i prefers to be a member of. The game is *additively separable* if there exists a *utility function* $v_i : N \rightarrow \mathbb{R}$ for each $i \in N$ such that

$$\forall S, T \in N_i : T \preceq_i S \Leftrightarrow \sum_{j \in T} v_i(j) \leq \sum_{j \in S} v_i(j) .$$

In an additively separable hedonic game the *payoff* $v_i(j)$ of player i for belonging to the same coalition as player j is independent of how other players are forming coalitions. Changing the value $v_i(i)$ has no effect on \preceq_i so we assume $v_i(i) = 0$.

1.2 An Example

We would like to give an example of an additively separable hedonic game. We will use biological terminology metaphorically to ease the understanding for the game. The game does *not* represent a serious attempt to model a biological system.

Assume that there are two buffaloes b_1 and b_2 in an area with n waterholes w_1, w_2, \dots, w_n . Each waterhole w_i has a capacity $c(w_i)$ specifying how much water a buffalo can drink from that hole per year. There are also two parasites p_1 and p_2 in the area. The only possible host for p_1 is b_1 and b_1 must drink a lot of water if p_1 is sitting on its back. The same goes for p_2 and b_2 . Now assume that b_1 and b_2 are enemies and that a buffalo must drink water corresponding to half the total capacity C of the waterholes if it is the host of a parasite. This system can be viewed as an additively separable hedonic game depicted as a weighted directed graph in Fig. 1 where the weight of edge (i, j) is $v_i(j)$ - if there is no edge (i, j) then $v_i(j) = 0$. We have added two edges (b_1, b_2) and (b_2, b_1) with capacity $-C - 1$ to model that b_1 and b_2 are enemies. Please note that the waterholes are also players in the game. The waterholes do not care to which coalitions they belong.

1.3 Stability

In this paper we will focus on one type of stability: *Nash stability*. We refer to [7] for more information on other types of stability.

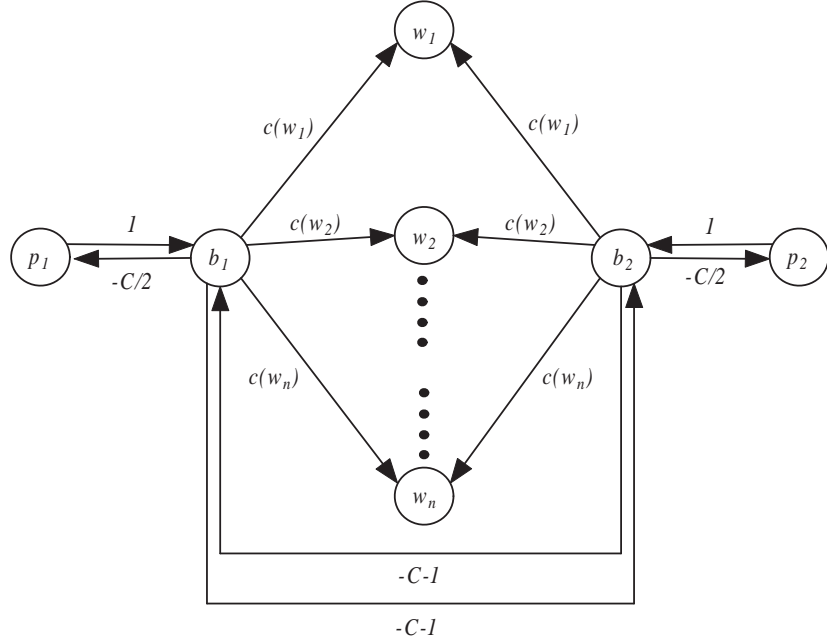


Fig. 1. An additively separable hedonic game.

A partition $\Pi = \{S_1, S_2, \dots, S_K\}$ of N is *Nash stable* if it is impossible to find a player who would be strictly better off if the player left his coalition and joined one of the other coalitions in the partition. If $S_\Pi(i)$ denotes the set in the partition Π such that $i \in S_\Pi(i)$ then Π is Nash stable if and only if

$$\forall i \in N, \forall S_k \in \Pi \cup \{\emptyset\} : S_k \cup \{i\} \preceq_i S_\Pi(i) .$$

Now consider the game in Fig. 1. A partition of the players is not Nash stable if b_1 is not the host of p_1 - in this case p_1 would be strictly better off by joining $S_\Pi(b_1)$. This fact can be expressed more formally: $S_\Pi(b_1) \cup \{p_1\} \succ_{p_1} S_\Pi(p_1)$ if $S_\Pi(p_1) \neq S_\Pi(b_1)$. As an exercise the reader is invited to figure out necessary and sufficient conditions for the existence of a Nash stable partition of this game. We will return to this problem in Sect. 2.

1.4 Related Work

Ballester has shown in [1] that the problem of deciding whether a Nash stable partition exists in a hedonic game with *arbitrary* preferences is NP-complete.

On the other hand Bogomolnaia and Jackson show in [8] that a Nash stable partition exists in every additively separable hedonic game with *symmetric* preferences. The preferences are symmetric if $\forall i, j \in N : v_i(j) = v_j(i)$. If v_{ij} is the common value for $v_i(j)$ and $v_j(i)$ in a symmetric game then Bogomolnaia

and Jackson show that any partition Π maximizing $f(\Pi) = \sum_{S \in \Pi} \sum_{i,j \in S} v_{ij}$ is Nash stable.

Flake et al. work with so called *communities* in [5] where the objective is to divide a network into clusters. The web graph and the CiteSeer graph are examples of networks that are processed in [5]. Using the terminology from coalition formation games a *community* is a subset of players $C \subseteq N$ in an additively separable game with symmetric preferences such that $\forall i \in C : \sum_{j \in C} v_{ij} \geq \sum_{j \in N-C} v_{ij}$. In other words each player in C gets at least half the total possible payoff by belonging to C . Flake et al. show that the problem of deciding whether it is possible to partition N into k communities is NP-complete. Such a partition is Nash stable but a Nash stable partition is not necessarily a partition into communities. The proof techniques used in this paper are similar to those used in [5].

Burani and Zwicker introduces the concept of *descending separable* preferences in [2]. Burani and Zwicker show that descending separable preferences guarantees the existence of a Nash stable partition. They also show that descending separable preferences do not imply and are not implied by additively separable preferences.

1.5 Our Results

In Sect. 2 we restrict our attention to additively separable hedonic games compared to Ballester [1]. Compared to Bogomolnaia and Jackson [8], we also allow asymmetric preferences. Informally we show that things are complicated even when looking at additively separable hedonic games. With an intuitively clear proof based on the example in Sect. 1.2 we show that the problem of deciding whether a Nash stable partition exists in a hedonic game remains NP-complete when restricting to additively separable preferences.

In Sect. 3 we show that *computing* Nash stable and individually stable partitions is hard in games with symmetric preferences. To be more specific we show that the problem of deciding whether a *non trivial* Nash stable partition exists in an additively separable hedonic game with *non-negative* and *symmetric* preferences is NP-complete. This result also applies to individually stable partitions since individually stable partitions are Nash stable and vice versa in such games.

2 Restricting to Additively Separable Games

We will now formally define the problem of deciding whether a Nash stable partition exists in an additively separable hedonic game:

Definition 1. *The ASHNASH problem:*

- *Instance:* A set $N = \{1, 2, \dots, n\}$ and a function $v_i : N \rightarrow \mathbb{R}$ such that $v_i(i) = 0$ for each $i \in N$.

– Question: Does a partition Π of N exist such that

$$\forall i \in N, \forall S_k \in \Pi \cup \{\emptyset\} : \sum_{j \in S_{\Pi}(i)} v_i(j) \geq \sum_{j \in S_k \cup \{i\}} v_i(j) . \quad (1)$$

We are now in a position to prove that this problem is intractable.

Theorem 1. *ASHNASH is NP-complete.*

Proof. It is easy to check in polynomial time that Π is a partition satisfying (1) thus ASHNASH is in NP.

We will transform an instance of the NP-complete problem PARTITION [6] into an instance of ASHNASH in polynomial time such that the answers to the questions posed in the two instances are identical - if such a transformation exists we will write PARTITION \propto ASHNASH following the notation in [6]. This means that we can solve the NP-complete problem PARTITION in polynomial time if we can solve ASHNASH in polynomial time thus ASHNASH is NP-complete since it is a member of NP. The rest of the proof explains the details of the transformation.

An instance² of PARTITION is a finite set $W = \{w_1, w_2, \dots, w_n\}$ and a capacity $c(w) \in \mathbb{Z}^+$ for each $w \in W$. The question is whether a subset $W' \subset W$ exists such that $\sum_{w \in W'} c(w) = \frac{C}{2}$ where $C = \sum_{w \in W} c(w)$.

Now suppose we are given an instance of PARTITION. The PARTITION instance is easily transformed into the buffalo-parasite-game from Sect. 1.2 in polynomial time. All we have to do to translate this as an ASHNASH instance is to perform a simple numbering of the players in the game.

Now we only have to show that a Nash stable partition of the game in Fig. 1 exists if and only if W' exists. This can be seen from the following argument:

- The partition $\Pi = \{\{b_1, p_1\} \cup W', \{b_2, p_2\} \cup W - W'\}$ is Nash stable if W' exists.
- Now assume that a Nash stable partition Π exists and define $W_1 = S_{\Pi}(b_1) \cap W$ and $W_2 = S_{\Pi}(b_2) \cap W$. The set $S_{\Pi}(b_1)$ must contain p_1 . Due to the stability we can conclude that $\sum_{w \in W_1} c(w) \geq \frac{C}{2}$ - otherwise b_1 would be better off by its own. By a symmetric argument we have $\sum_{w \in W_2} c(w) \geq \frac{C}{2}$. The two nodes b_1 and b_2 are not in the same coalition so the two sets W_1 and W_2 are disjoint, so we can conclude that $\sum_{w \in W_1} c(w) = \sum_{w \in W_2} c(w) = \frac{C}{2}$. We can take $W' = W_1$.

□

3 Non-negative and Symmetric Preferences

In this section we will restrict our attention to additively separable games with *non-negative* and *symmetric* preferences. The trivial partition where all players

² The objects constituting an instance in [6] are renamed to match the example in Sect. 1.2

cooperate is optimal for all players in such games. If all players form a clique where the payoffs are identical then the trivial partition is the only Nash stable partition.

Now let us on the other hand assume that two disjoint communities S and T of players exist as defined in Sect. 1.4. If we collapse these communities to two players s and t then we can effectively calculate the s - t minimum cut in the underlying graph for the game. This cut defines a non trivial Nash stable partition. We will denote a non trivial Nash stable partition as an *inefficient equilibrium*. In this section we will prove that inefficient equilibriums generally are hard to compute. To be more specific we will prove that the problem of deciding whether they exist is NP-complete.

As in the proof of Theorem 1 we need a known NP-complete problem in the proof of the theorem of this section. The “base” problem of the proof in this section is the following problem:

Definition 2. EQUAL CARDINALITY PARTITION

- Instance: A finite set $W = \{w_1, w_2, \dots, w_n\}$ and a capacity $c(w) \in \mathbb{Z}^+$ for each $w \in W$
- Question: Does a non trivial partition of W exist such that $|W_i| = |W_j|$ and $\sum_{w \in W_i} c(w) = \sum_{w \in W_j} c(w)$ for all sets W_i and W_j in the partition?

EQUAL CARDINALITY PARTITION is closely related to the balanced version of PARTITION where we are looking for a set $W' \subset W$ such that $\sum_{w \in W'} c(w) = \frac{C}{2}$ and $|W'| = \frac{|W|}{2}$. The balanced version of PARTITION is known to be NP-complete [6]. An instance of the balanced version of PARTITION is transformed into an equivalent instance of EQUAL CARDINALITY PARTITION by adding two more elements to the set W - both with capacity $C + 1$. This shows that EQUAL CARDINALITY PARTITION is NP-complete since it is easily seen to belong to NP.

We will now formally define the problem of deciding whether a non trivial Nash stable partition exists in an additively separable hedonic game with non-negative and symmetric preferences:

Definition 3. The INEFFICIENT EQUILIBRIUM problem:

- Instance: A set $N = \{1, 2, \dots, n\}$ and a function $v_i : N \rightarrow \mathbb{R}^+ \cup \{0\}$ such that $v_i(i) = 0$ for each $i \in N$ and $v_i(j) = v_j(i)$ for each $i, j \in N$.
- Question: Does a non trivial partition Π of N exist such that

$$\forall i \in N, \forall S_k \in \Pi \cup \{\emptyset\} : \sum_{j \in S_{\Pi}(i)} v_i(j) \geq \sum_{j \in S_k \cup \{i\}} v_i(j) .$$

Theorem 2. INEFFICIENT EQUILIBRIUM is NP-complete.

Proof. We will show that EQUAL CARDINALITY PARTITION \propto INEFFICIENT EQUILIBRIUM. By the same line of reasoning as in the proof of Theorem 1 we conclude that INEFFICIENT EQUILIBRIUM is NP-complete since INEFFICIENT EQUILIBRIUM is easily seen to belong to NP.

We will now show how to transform an instance of EQUAL CARDINALITY PARTITION into an equivalent instance of INEFFICIENT EQUILIBRIUM. All the members of W are players in the instance of INEFFICIENT EQUILIBRIUM and the payoff for w_i and w_j for cooperating is $c(w_i) + c(w_j) + C$. For each player w_i we also add a player z_i . Player z_i only gets a strictly positive payoff by cooperating with w_i - in this case the payoff is $2c(w_i) + C$. Figure 2 depicts a part of the INEFFICIENT EQUILIBRIUM instance as an undirected weighted graph. The members of W are fully connected but z_i is only connected to w_i in the graph.

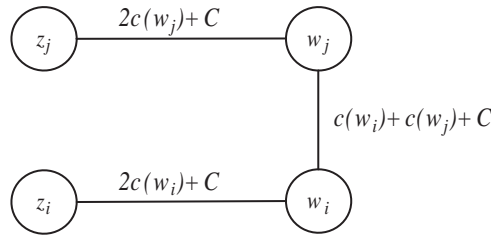


Fig. 2. A part of a game with positive and symmetric preferences.

We will now prove that the two instances are equivalent:

- Suppose that we have a non trivial Nash stable partition Π of the players in Fig. 2. For $S_k \in \Pi$ we define $W_k = S_k \cap W$. The player z_i cooperates with w_i - otherwise Π would not be stable. The total payoff of $w_i \in W_k$ is $|W_k|(C + c(w_i)) + \sum_{w \in W_k} c(w)$.
 - $|W_i| = |W_j|$: If $|W_i| < |W_j|$ then all the players in W_i would be strictly better off by joining W_j . This contradicts that Π is stable.
 - $\sum_{w \in W_i} c(w) = \sum_{w \in W_j} c(w)$: Now assume $\sum_{w \in W_i} c(w) < \sum_{w \in W_j} c(w)$. Once again the players in W_i would be strictly better off by joining W_j since $|W_i| = |W_j|$. Yet another contradiction.
- Suppose that we have a non trivial partition of W into sets with equal cardinality and capacity. For a set W_i in this partition let S_i be the union of W_i and the corresponding z -members. The set of S_i 's is easily seen to be a non trivial Nash stable partition of the game in Fig. 2.

□

4 Conclusion

We have shown that the problem of deciding whether a Nash stable partition exists in an additively separable hedonic game is NP-complete. For additively separable games with non-negative and symmetric preferences we have shown that the problem of deciding whether a non trivial Nash stable partition exists is NP-complete.

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