

# Event Structures, Stable Families and Concurrent Games

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## Preface

These rather uneven lecture notes introduce a theory of two-party games still under development. A lot can be said for a general theory to unify all manner of games found in the literature. But this has not been the main motivation. That has been the development of a generalized domain theory, to lift the methodology of domain theory and denotational semantics to address the highly interactive nature of computation we find today. There are several arguments why the next generation of domain theory should be an intensional theory, one which pays careful attention to the ways in which output is computed from input. One is that if the theory is to be able to reason about operational concerns it had better address them, albeit abstractly. Another is that sometimes the demands of compositionality force denotations to be more intensional than one would at first expect; this occurs for example with nondeterministic dataflow—see the Introduction. These notes take seriously the idea that intensional aspects be described by strategies, and to fit computational needs adequately try to understand the concept of strategy very broadly.

This idea comes from game semantics where the domains and continuous functions of traditional domain theory and denotational semantics are replaced by games and strategies. Strategies supercede functions because they give a much better account of interaction extended in time. (Functions, if you like, have too clean a separation of interaction into input and output.) In traditional denotational semantics a program phrase or process term denotes a continuous function, whereas in game semantics a program phrase or process term denotes a strategy. However, traditional game semantics is not always general enough, for instance in accounting for nondeterministic or concurrent computation. Rather than extending traditional game semantics with various bells and whistles, these notes attempt to carve out a general theory of games within a general model of nondeterministic, concurrent computation. The model chosen is the partial-order model of event structures, and for technical reasons, its enlargement to stable families. Event structures have the advantage of occupying a central position within models for concurrency, and the development here should suggest analogous developments for other ‘partial-order’ models such as Mazurkiewicz trace languages, Petri nets and asynchronous transition systems, and even ‘interleaving’ models based on transition systems or sequences.

In their present state, these notes are inadequate in several ways. First, they don’t account for games with back-tracking, games where play can revisit previous positions. While a little odd from the point of view of everyday games, this feature is very important in game semantics for instance in order to re-evaluate the argument to a function.<sup>1</sup> Second, the notes don’t have enough examples. Third, the notes say almost nothing on the *uses* of games and strategies in semantics, types, logic and verification. I hope to some extent to make up for these inadequacies in the lectures and in the mini-projects, perhaps too on the

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<sup>1</sup>The theory is being extended to allow back-tracking and copying via event structures with symmetry, which support a rich variety of pseudo (co)monads to achieve this.

course webpage. What I claim the notes do do, is begin to unify a variety of approaches and provide canonical general constructions and results, which leave the student better placed to structure and analyse critically the often arcane world of games and strategies in the literature—I hope the mini-projects will demonstrate this!



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# Chapter 1

## Introduction

Games and strategies are everywhere, in logic, philosophy, computer science, economics, in leisure and in life.

*Slogan for this course: Processes are nondeterministic concurrent strategies.*

### 1.1 Motivation

We summarise some reasons for developing a theory of nondeterministic concurrent games and strategies.

#### 1.1.1 What is a process?

In the earliest days of computer science it became accepted that a computation was essentially an (effective) partial function  $f : \mathbb{N} \rightarrow \mathbb{N}$  between the natural numbers. This view underpins the Church-Turing thesis on the universality of computability.

As computer science matured it demanded increasingly sophisticated mathematical representations of processes. The pioneering work of Strachey and Scott in the denotational semantics of programs assumed a view of a process still as a function  $f : D \rightarrow D'$ , but now acting in a continuous fashion between datatypes represented as special topological spaces, ‘domains’  $D$  and  $D'$ ; reflecting the fact that computers can act on complicated, conceptually-infinite objects, but only by virtue of their finite approximations.

In the 1960’s, around the time that Strachey started the programme of denotational semantics, Petri advocated his radical view of a process, expressed in terms of its events and their effect on local states—a model which addressed directly the potentially distributed nature of computation, but which, in common with many other current models, ignored the distinction between data and process implicit in regarding a process as a function. Here it seems that an adequate notion of process requires a marriage of Petri’s view of a process and

the vision of Scott and Strachey. An early hint in this direction came in answer to the following question.

What is the information order in domains? There are essentially two answers in the literature, the ‘*topological*,’ the most well-known from Scott’s work, and the ‘*temporal*,’ arising from the work of Berry:

- *Topological*: the basic units of information are *propositions* describing finite properties; more information corresponds to more propositions being true. Functions are ordered pointwise.
- *Temporal*: the basic units of information are *events*; more information corresponds to more events having occurred over time. Functions are restricted to ‘stable’ functions and ordered by the intensional ‘stable order,’ in which common output has to be produced for the same minimal input. Berry’s specialized domains ‘dI-domains’ are represented by event structures.

In truth, Berry developed ‘stable domain theory’ by a careful study of how to obtain a suitable category of domains with stable rather than all continuous functions. He arrived at the axioms for his ‘dI-domains’ because he wanted function spaces (so a cartesian-closed category). The realization that dI-domains were precisely those domains which could be represented by event structures, came a little later.

### 1.1.2 From models for concurrency

Causal models are alternatively described as: causal-dependence models; independence models; non-interleaving models; true-concurrency models; and partial-order models. They include Petri nets, event structures, Mazurkiewicz trace languages, transition systems with independence, multiset rewriting, and many more. The models share the central feature that they represent processes in terms of the events they can perform, and that they make explicit the causal dependency and conflicts between events.

Causal models have arisen, and have sometimes been rediscovered as *the* natural model, in many diverse and often unexpected areas of application:

*Security protocols*: for example, forms of event structure, strand spaces, support reasoning about secrecy and authentication through causal relations and the freshness of names;

*Systems biology*: ideas from Petri nets and event structures are used in taming the state-explosion in the stochastic simulation of biochemical processes and in the analysis of biochemical pathways;

*Hardware*: in the design and analysis of asynchronous circuits;

*Types and proof*: event structures appear as representations of propositions as types, and of proofs;

*Nondeterministic dataflow*: where numerous researchers have used or rediscovered causal models in providing a compositional semantics to nondeterministic dataflow;

*Network diagnostics*: in the patching together local of fault diagnoses of com-

munication networks;

*Logic of programs*: in concurrent separation logic where artificialities in Brookes' pioneering soundness proof are obviated through a Petri-net model;

*Partial order model checking*: following the seminal work of McMillan the unfolding of Petri nets (described below) is exploited in recent automated analysis of systems;

*Distributed computation*: event structures appear both classically, *e.g.* in early work of Lamport, and recently in the Bayesian analysis of trust and modelling multicore memory.

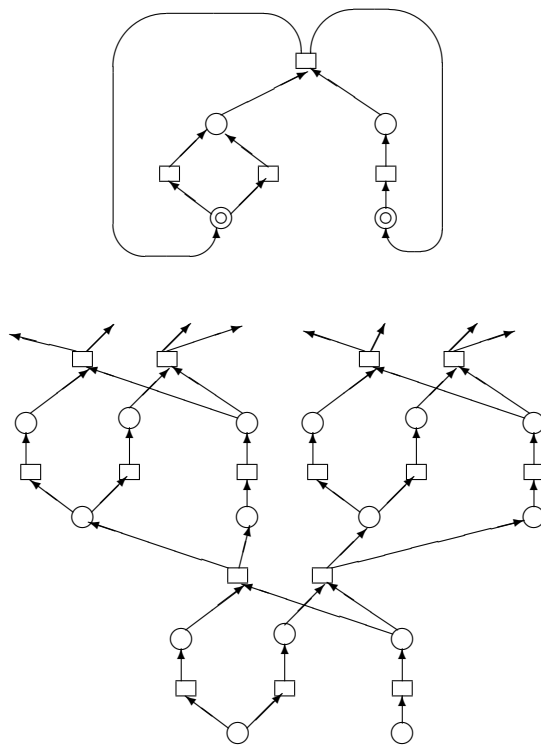
To illustrate the close relationship between Petri nets and the 'partial-order models' of occurrence nets and event structures, we sketch how a (1-safe) Petri net can be unfolded first to a net of occurrences and from there to an event structure [1]. The unfolding construction is analogous to the well-known method of unfolding a transition system to a tree, and is central to several analysis tools in the applications above. In the figure, the net on top has loops. The net below it is its *occurrence-net unfolding*. It consists of all the occurrences of conditions and events of the original net, and is infinite because of the original repetitive behaviour. The occurrences keep track of what enabled them. The simplest form of event structure, the one we shall consider here, arises by abstracting away the conditions in the occurrence net and capturing their role in relations of causal dependency and conflict on event occurrences.

The relations between the different forms of causal models are well understood [2]. Despite this and their often very successful, specialized applications, causal models lack a *comprehensive* theory which would support their systematic use in giving semantics to a broad range of programming and process languages, in particular we lack an expressive form of '*domain theory*' for causal models with rich higher-order type constructions needed by mathematical semantics.

### 1.1.3 From semantics

Denotational semantics and domain theory of Scott and Strachey set the standard for semantics of computation. The theory provided a global mathematical setting for sequential computation, and thereby placed programming languages in connection with each other; connected with the mathematical worlds of algebra, topology and logic; and inspired programming languages, type disciplines and methods of reasoning. Despite the many striking successes it has become very clear that many aspects of computation do not fit within the traditional framework of denotational semantics and domain theory. In particular, classical domain theory has not scaled up to the more intricate models used in interactive/distributed computation. Nor has it been as operationally informative as one could hope.

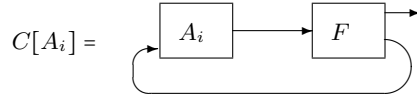
While, as Kahn was early to show, deterministic dataflow is a shining application of simple domain theory, nondeterministic dataflow is beyond its scope. The compositional semantics of nondeterministic dataflow needs a form of generalized relation which specifies the *ways* input-output pairs are realized. A com-



**A Petri net and its occurrence-net unfolding**

elling example comes from the early work of Brock and Ackerman who were the first to emphasize the difficulties in giving a compositional semantics to non-deterministic dataflow, though our example is based on simplifications in the later work of Rabinovich and Trakhtenbrot, and Russell.

### Nondeterministic dataflow—Brock-Ackerman anomaly



There are two simple nondeterministic processes  $A_1$  and  $A_2$ , which have the same input-output relation, and yet behave differently in the common feedback context  $C[-]$ , illustrated above. The context consists of a fork process  $F$  (a process that copies every input to two outputs), through which the output of the automata  $A_i$  is fed back to the input channel, as shown in the figure. Process  $A_1$  has a choice between two behaviours: either it outputs a token and stops, *or* it outputs a token, waits for a token on input and then outputs another token. Process  $A_2$  has a similar nondeterministic behaviour: Either it outputs a token and stops, *or* it waits for an input token, then outputs two tokens. For both automata, the input-output relation relates empty input to the eventual output of one token, and non-empty input to one or two output tokens. But  $C[A_1]$  can output two tokens, whereas  $C[A_2]$  can only output a single token. Notice that  $A_1$  has two ways to realize the output of a single token from empty input, while  $A_2$  only has one. It is this extra way, not caught in a simple input-output relation, that gives  $A_1$  the richer behaviour in the feedback context.

Over the years there have been many solutions to giving a compositional semantics to nondeterministic dataflow. But they all hinge on some form of generalized relation, to distinguish the different ways in which output is produced from input. A compositional semantics can be given using *stable spans* of event structures, an extension of Berry’s stable functions to include nondeterminism [3]—see Section 6.2.1.

How are we to extend the methodology of denotational semantics to the much broader forms of computational processes we need to design, understand and analyze today? How are we to maintain clean algebraic structure and abstraction alongside the operational nature of computation?

Game semantics advanced the idea of replacing the traditional continuous functions of domain theory and denotational semantics by strategies. The reason for doing this was to obtain a representation of interaction in computation that was more faithful to operational reality. It is not always convenient or mathematically tractable to assume that the environment interacts with a computation in the form of an input argument. It is built into the view of a process as a strategy that the environment can direct the course of evolution of a process throughout its duration. Game semantics has had many dramatic successes. But it has developed from simple well-understood games, based on alternating

sequences of player and opponent moves, to sometimes arcane extensions and generalizations designed to fit the demands of a succession of additional programming or process features. It is perhaps time to stand back and see how games fit within a very general model of computation, to understand better what current features of games in computer science are simply artefacts of the particular history of their development.

#### 1.1.4 From logic

An informal understanding of games and strategies goes back at least as far as the ancient Greeks where truth was sought through debate using the dialectic method; a contention being true if there was an argument for it that could survive all counter-arguments. Formalizing this idea, logicians such as Lorenzen and Blass investigated the meaning of a logical assertion through strategies in a game built up from the assertion. These ideas were reinforced in game semantics which can provide semantics to proofs as well as programs. The study of the mathematics and computational nature of proof continues. There are several strands of motivation for games in logic. Along with automata games constitute one of the tools of logic and algorithmics. Games are used in verification and, for example, the central equivalence of bisimulation on processes has a reading in terms of strategies.

# Chapter 2

## Event structures

Event structures are a fundamental model of concurrent computation and, along with their extension to stable families, provide a mathematical foundation for the course.

### 2.1 Event structures

Event structures are a model of computational processes. They represent a process, or system, as a set of event occurrences with relations to express how events causally depend on others, or exclude other events from occurring. In one of their simpler forms they consist of a set of events on which there is a consistency relation expressing when events can occur together in a history and a partial order of causal dependency—writing  $e' \leq e$  if the occurrence of  $e$  depends on the previous occurrence of  $e'$ .

An *event structure* comprises  $(E, \leq, \text{Con})$ , consisting of a set  $E$ , of *events* which are partially ordered by  $\leq$ , the *causal dependency relation*, and a nonempty *consistency relation*  $\text{Con}$  consisting of finite subsets of  $E$ , which satisfy

$$\begin{aligned} \{e' \mid e' \leq e\} &\text{ is finite for all } e \in E, \\ \{e\} &\in \text{Con for all } e \in E, \\ Y \subseteq X \in \text{Con} &\implies Y \in \text{Con}, \text{ and} \\ X \in \text{Con} \ \& \ e \leq e' \in X &\implies X \cup \{e\} \in \text{Con}. \end{aligned}$$

The events are to be thought of as event occurrences without significant duration; in any history an event is to appear at most once. We say that events  $e, e'$  are *concurrent*, and write  $e \text{ co } e'$  if  $\{e, e'\} \in \text{Con}$  &  $e \not\leq e'$  &  $e' \not\leq e$ . Concurrent events can occur together, independently of each other. The relation of *immediate* dependency  $e \rightarrow e'$  means  $e$  and  $e'$  are distinct with  $e \leq e'$  and no event in between.

An event structure represents a process. A configuration is the set of all events which may have occurred by some stage, or history, in the evolution of

the process. According to our understanding of the consistency relation and causal dependency relations a configuration should be consistent and such that if an event appears in a configuration then so do all the events on which it causally depends.

The *configurations*,  $\mathcal{C}(E)$ , of an event structure  $E$  consist of those subsets  $x \subseteq E$  which are

*Consistent:*  $\forall X \subseteq x. X \text{ is finite} \Rightarrow X \in \text{Con}$ , and

*Down-closed:*  $\forall e, e'. e' \leq e \in x \implies e' \in x$ .

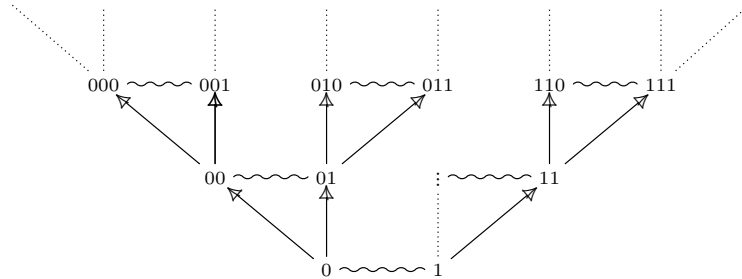
We shall largely work with *finite* configurations, written  $\mathcal{C}(E)$ . Write  $\mathcal{C}^\infty(E)$  for the set of *finite and infinite* configurations of the event structure  $E$ .

The configurations of an event structure are ordered by inclusion, where  $x \subseteq x'$ , *i.e.*  $x$  is a sub-configuration of  $x'$ , means that  $x$  is a sub-history of  $x'$ . Note that an individual configuration inherits an order of causal dependency on its events from the event structure so that the history of a process is captured through a partial order of events. The finite configurations correspond to those events which have occurred by some finite stage in the evolution of the process, and so describe the possible (finite) states of the process.

For  $X \subseteq E$  we write  $[X]$  for  $\{e \in E \mid \exists e' \in X. e \leq e'\}$ , the down-closure of  $X$ . The axioms on the consistency relation ensure that the down-closure of any finite set in the consistency relation is a finite configuration, and that any event appears in a configuration: given  $X \in \text{Con}$  its down-closure  $\{e' \in E \mid \exists e \in X. e' \leq e\}$  is a finite configuration; in particular, for an event  $e$ , the set  $[e] =_{\text{def}} \{e' \in E \mid e' \leq e\}$  is a configuration describing the whole causal history of the event  $e$ .

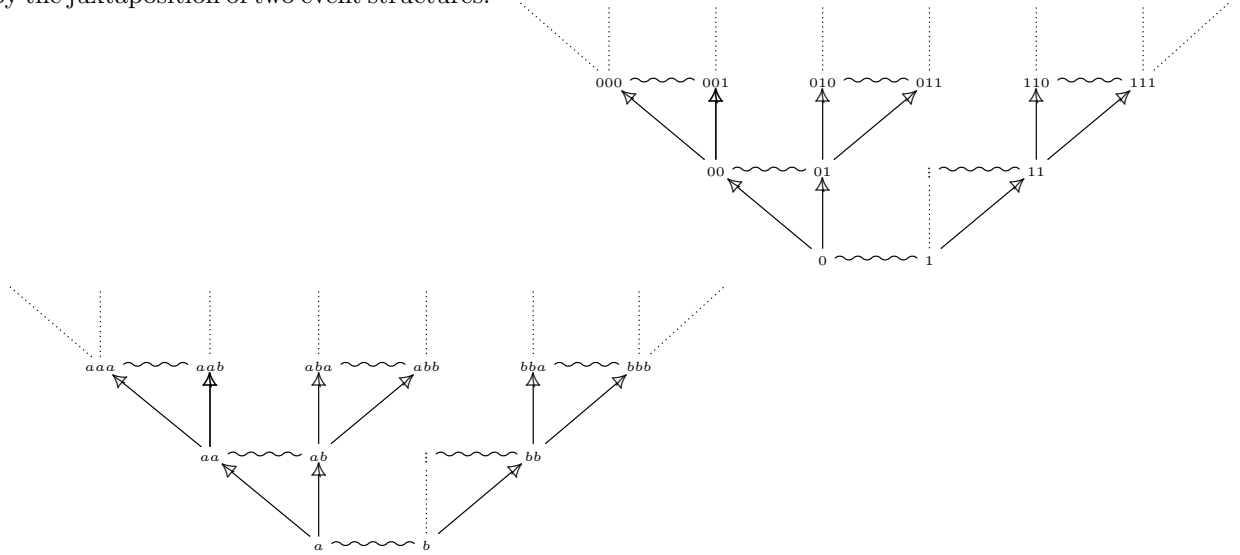
When the consistency relation is determined by the pairwise consistency of events we can replace it by a binary relation or, as is more usual, by a complementary binary conflict relation on events (written as  $\#$  or  $\smile$ ).

*Example 1.* The diagram below illustrates an event structure representing streams of 0s and 1s:



Above we have indicated conflict (or inconsistency) between events by  $\smile$ . The event structure representing pairs of 0/1-streams and  $a/b$ -streams is represented

by the juxtaposition of two event structures:



*Exercise 2.* Draw the event structure of the occurrence net unfolding in the introduction. □

### 2.1.1 Maps of event structures

Let  $E$  and  $E'$  be event structures. A (*partial*) *map* of event structures  $f : E \rightarrow E'$  is a partial function on events  $f : E \rightarrow E'$  such that for all  $x \in \mathcal{C}(E)$  its direct image  $fx \in \mathcal{C}(E')$  and

$$\text{if } e_1, e_2 \in x \text{ and } f(e_1) = f(e_2) \text{ (with both defined), then } e_1 = e_2.$$

The map expresses how the occurrence of an event  $e$  in  $E$  induces the coincident occurrence of the event  $f(e)$  in  $E'$  whenever it is defined. The map  $f$  respects the instantaneous nature of events: two distinct event occurrences which are consistent with each other cannot both coincide with the occurrence of a common event in the image. Partial maps of event structures compose as partial functions, with identity maps given by identity functions.

For any event  $e$  a map of event structures  $f : E \rightarrow E'$  must send the configuration  $[e]$  to the configuration  $f[e]$ . Consequently the map  $f$  reflects causal dependency: whenever  $f(e)$  and  $f(e')$  are both defined with  $f(e') \leq f(e)$ , then  $e' \leq e$ . It follows that partial maps preserve the concurrency relation, when defined.

We will say the map is *total* if the function  $f$  is total. Notice that for a total map  $f$  the condition on maps now says it is *locally injective*, in the sense that w.r.t. any configuration  $x$  of the domain the restriction of  $f$  to a function from  $x$  is injective; the restriction of  $f$  to a function from  $x$  to  $fx$  is thus bijective. Say a total map of event structures is *rigid* when it preserves causal dependency.

**Definition 3.** Write  $\mathcal{E}$  for the category of event structures with (partial) maps. Write  $\mathcal{E}_t$  for the category of event structures with total maps.

*Exercise 4.* Show a map  $f : A \rightarrow B$  of  $\mathcal{E}$  is mono iff the function  $\mathcal{C}(A) \rightarrow \mathcal{C}(B)$  taking configuration  $x$  to its direct image  $fx$  is injective. [Recall a map  $f : A \rightarrow B$  is mono iff for all maps  $g, h : C \rightarrow A$  if  $fg = fh$  then  $g = h$ .]  $\square$

**Proposition 5.** Let  $E$  and  $E'$  be event structures. Suppose

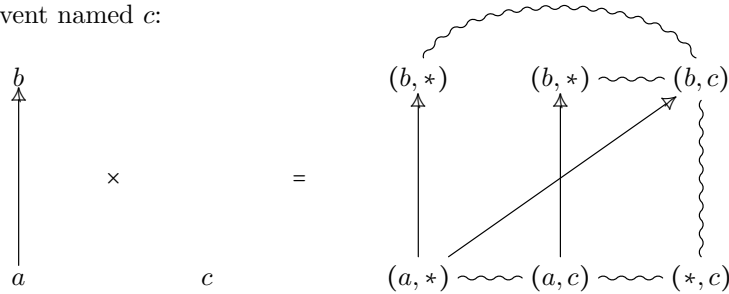
$$\theta_x : x \cong \theta_x x, \text{ indexed by } x \in \mathcal{C}(E),$$

is a family of bijections such that whenever  $\theta_y : y \cong \theta_y y$  is in the family then its restriction  $\theta_z : z \cong \theta_z z$  is also in the family, whenever  $z \in \mathcal{C}(E)$  and  $z \subseteq y$ . Then,  $\theta =_{\text{def}} \bigcup_{x \in \mathcal{C}(E)} \theta_x$  is the unique total map of event structures from  $E$  to  $E'$  such that  $\theta x = \theta_x x$  for all  $x \in \mathcal{C}(E)$ .

*Proof.* The conditions ensure that  $\theta =_{\text{def}} \bigcup_{x \in \mathcal{C}(A)} \theta_x$  is a function  $\theta : A \rightarrow B$  such that the image of any finite configuration  $x$  of  $A$  under  $\theta$  is a configuration of  $B$  and local injectivity holds.  $\square$

## 2.2 Products of event structures

The category of event structures has products, which essentially allow arbitrary synchronizations between their components. For example, here is an illustration of the product of two event structures  $a \rightarrow b$  and  $c$ , the later comprising just a single event named  $c$ :



The original event  $b$  has split into three events, one a synchronization with  $c$ , another  $b$  occurring unsynchronized after an unsynchronized  $a$ , and the third  $b$  occurring unsynchronized after  $a$  synchronizes with  $c$ . The splittings correspond to the different histories of the event.

It can be awkward to describe operations such as products, pullbacks and synchronized parallel compositions directly on the simple event structures here, essentially because an event determines its whole causal history. One closely related and more versatile, though perhaps less intuitive and familiar, model is that of stable families. Stable families will play an important technical role in establishing and reasoning about constructions on event structures.

# Chapter 3

## Stable families

Stable families, their basic properties and relations to event structures are developed.<sup>1</sup>

### 3.1 Stable families

The notion of stable family extends that of finite configurations of an event structure to allow an event can occur in several incompatible ways.

*Notation 6.* Let  $\mathcal{F}$  be a family of subsets. Let  $X \subseteq \mathcal{F}$ . We write  $X \uparrow$  for  $\exists y \in \mathcal{F}. \forall x \in X. x \subseteq y$  and say  $X$  is *compatible*. When  $x, y \in \mathcal{F}$  we write  $x \uparrow y$  for  $\{x, y\} \uparrow$ .

A *stable family* comprises  $\mathcal{F}$ , a nonempty family of finite subsets, satisfying:

*Completeness:*  $\forall Z \subseteq \mathcal{F}. Z \uparrow \implies \bigcup Z \in \mathcal{F}$ ;

*Stability:*  $\forall Z \subseteq \mathcal{F}. Z \neq \emptyset \ \& \ Z \uparrow \implies \bigcap Z \in \mathcal{F}$ ;

*Coincidence-freeness:* For all  $x \in \mathcal{F}$ ,  $e, e' \in x$  with  $e \neq e'$ ,

$$\exists y \in \mathcal{F}. y \subseteq x \ \& \ (e \in y \iff e' \notin y).$$

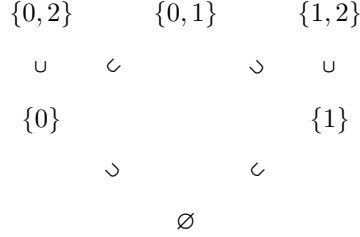
**Proposition 7.** *The family of finite configurations of an event structure forms a stable family.*

On the other hand stable families are more general than finite configurations of an event structure, as the following example shows.

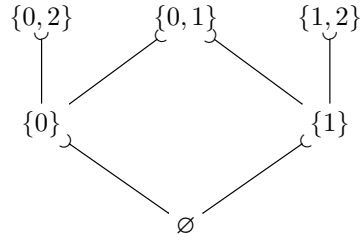
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<sup>1</sup>A useful reference for stable families is the report “Event structure semantics for CCS and related languages,” a full version of the article [4], available from [www.cl.cam.ac.uk/~gw104](http://www.cl.cam.ac.uk/~gw104), though its terminology can differ from that here.

*Example 8.* Let  $F$  be the stable family, with events  $E = \{0, 1, 2\}$ ,



or equivalently



where  $—\subset$  is the covering relation representing an occurrence of one event. The events 0 and 1 are concurrent, neither depends on the occurrence or non-occurrence of the other to occur. The event 2 can occur in two incompatible ways, either through event 0 having occurred or event 1 having occurred. This possibility can make stable families more flexible to work with than event structures.

A (partial) map of stable families  $f : \mathcal{F} \rightarrow \mathcal{G}$  is a partial function  $f$  from the events of  $\mathcal{F}$  to the events of  $\mathcal{G}$  such that for all  $x \in \mathcal{F}$ ,

$$f x \in \mathcal{G} \ \& \ (\forall e_1, e_2 \in x. f(e_1) = f(e_2) \implies e_1 = e_2).$$

Maps of stable families compose as partial functions, with identity maps given by identity functions.

**Proposition 9.** *Let  $f : \mathcal{F} \rightarrow \mathcal{G}$  be a map of stable families. Let  $e, e' \in x$ , a configuration of  $\mathcal{F}$ . If  $f(e)$  and  $f(e')$  are defined and  $f(e) \leq_{f x} f(e')$  then  $e \leq_x e'$ .*

**Definition 10.** Let  $\mathcal{F}$  be a stable family. We use  $x \text{---}\subset y$  to mean  $y$  covers  $x$  in  $\mathcal{F}$ , i.e.  $x \subset y$  in  $\mathcal{F}$  with nothing in between, and  $x \text{---}^e \subset y$  to mean  $x \cup \{e\} = y$  for  $x, y \in \mathcal{F}$  and event  $e \notin x$ . We sometimes use  $x \text{---}^e \subset$ , expressing that event  $e$  is enabled at configuration  $x$ , when  $x \text{---}^e \subset y$  for some  $y$ .

*Exercise 11.* Let  $\mathcal{F}$  be a nonempty family of sets satisfying the Completeness axiom in the definition of stable families. Show  $\mathcal{F}$  is coincidence-free iff

$$\forall x, y \in \mathcal{F}. x \not\sqsubset y \implies \exists x_1, e_1. x \text{---}^{e_1} \subset x_1 \subseteq y.$$

[Hint: For ‘only if’ use induction on the size of  $y \setminus x$ .]

□

### 3.1.1 Stable families and event structures

Finite configurations of an event structure form a stable family. Conversely, a stable family determines an event structure:

**Proposition 12.** *Let  $x$  be a configuration of a stable family  $\mathcal{F}$ . For  $e, e' \in x$  define*

$$e' \leq_x e \text{ iff } \forall y \in \mathcal{F}. y \subseteq x \ \& \ e \in y \implies e' \in y.$$

When  $e \in x$  define the prime configuration

$$[e]_x = \bigcap \{y \in \mathcal{F} \mid y \subseteq x \ \& \ e \in y\}.$$

Then  $\leq_x$  is a partial order and  $[e]_x$  is a configuration such that

$$[e]_x = \{e' \in x \mid e' \leq_x e\}.$$

Moreover the configurations  $y \subseteq x$  are exactly the down-closed subsets of  $\leq_x$ .

**Proposition 13.** *Let  $\mathcal{F}$  be a stable family. Then,  $\text{Pr}(\mathcal{F}) =_{\text{def}} (P, \text{Con}, \leq)$  is an event structure where:*

$$\begin{aligned} P &= \{[e]_x \mid e \in x \ \& \ x \in \mathcal{F}\}, \\ Z \in \text{Con} &\text{ iff } Z \subseteq P \ \& \ \bigcup Z \in \mathcal{F} \text{ and,} \\ p \leq p' &\text{ iff } p, p' \in P \ \& \ p \subseteq p'. \end{aligned}$$

*Exercise 14.* Prove the two propositions 12 and 13. □

The operation  $\text{Pr}$  is right adjoint to the “inclusion” functor, taking an event structure  $E$  to the stable family  $\mathcal{C}(E)$ . The unit of the adjunction  $E \rightarrow \text{Pr}(\mathcal{C}(E))$  takes an event  $e$  to the prime configuration  $[e] =_{\text{def}} \{e' \in E \mid e' \leq e\}$ . The counit  $\text{max} : \mathcal{C}(\text{Pr}(\mathcal{F})) \rightarrow \mathcal{F}$  takes prime configuration  $[e]_x$  to  $e$ .

**Definition 15.** Let  $\mathcal{F}$  be a stable family. W.r.t.  $x \in \mathcal{F}$ , write  $[e]_x =_{\text{def}} \{e' \in E \mid e' \leq_x e \ \& \ e' \neq e\}$ . The relation of *immediate* dependence of event structures generalizes: with respect to  $x \in \mathcal{F}$ : the relation  $e \rightarrow_x e'$  means  $e \leq_x e'$  with  $e \neq e'$  and no event in between.

## 3.2 Infinite configurations

We can extend a stable family to include infinite configurations, by constructing its “ideal completion.”

**Definition 16.** Let  $\mathcal{F}$  be a stable family. Define  $\mathcal{F}^\infty$  to comprise all  $\bigcup I$  where  $I \subseteq \mathcal{F}$  is an ideal (*i.e.*,  $I$  is a nonempty subset of  $\mathcal{F}$  closed downwards w.r.t.  $\subseteq$  in  $\mathcal{F}$  and such that if  $x, y \in I$  then  $x \cup y \in I$ ).

*Exercise 17.* For an event structure  $E$ , show  $\mathcal{C}^\infty(E) = \mathcal{C}(E)^\infty$ . □

*Exercise 18.* Let  $\mathcal{F}$  be a stable family. Show  $\mathcal{F}^\infty$  satisfies:

*Completeness:*  $\forall Z \subseteq \mathcal{F}^\infty. Z \uparrow \implies \bigcup Z \in \mathcal{F}^\infty$ ;  
*Stability:*  $\forall Z \subseteq \mathcal{F}^\infty. Z \neq \emptyset \ \& \ Z \uparrow \implies \bigcap Z \in \mathcal{F}^\infty$ ;  
*Coincidence-freeness:* For all  $x \in \mathcal{F}^\infty$ ,  $e, e' \in x$  with  $e \neq e'$ ,  

$$\exists y \in \mathcal{F}^\infty. y \subseteq x \ \& \ (e \in y \iff e' \notin y)$$
;  
*Finiteness:* For all  $x \in \mathcal{F}^\infty$ ,

$$\forall e \in x \exists y \in \mathcal{F}. e \in y \ \& \ y \subseteq x \ \& \ y \text{ is finite .}$$

Show that  $\mathcal{F}$  consists of precisely the finite sets in  $\mathcal{F}^\infty$ . □

### 3.3 Process constructions

#### 3.3.1 Products

Let  $\mathcal{A}$  and  $\mathcal{B}$  be stable families with events  $A$  and  $B$ , respectively. Their product, the stable family  $\mathcal{A} \times \mathcal{B}$ , has events comprising pairs in  $A \times_* B =_{\text{def}} \{(a, *) \mid a \in A\} \cup \{(a, b) \mid a \in A \ \& \ b \in B\} \cup \{(*, b) \mid b \in B\}$ , the product of sets with partial functions, with (partial) projections  $\pi_1$  and  $\pi_2$ —treating  $*$  as ‘undefined’—with configurations

$$\begin{aligned} x \in \mathcal{A} \times \mathcal{B} \text{ iff} \\ x \text{ is a finite subset of } A \times_* B \text{ such that } \pi_1 x \in \mathcal{A} \ \& \ \pi_2 x \in \mathcal{B}, \\ \forall e, e' \in x. \pi_1(e) = \pi_1(e') \text{ or } \pi_2(e) = \pi_2(e') \implies e = e', \ \& \\ \forall e, e' \in x. e \neq e' \implies \exists y \subseteq x. \pi_1 y \in \mathcal{A} \ \& \ \pi_2 y \in \mathcal{B} \ \& \\ (e \in y \iff e' \notin y). \end{aligned}$$

Right adjoints preserve products. Consequently we obtain a product of event structures  $A$  and  $B$  by first regarding them as stable families  $\mathcal{C}(A)$  and  $\mathcal{C}(B)$ , forming their product  $\mathcal{C}(A) \times \mathcal{C}(B)$ ,  $\pi_1, \pi_2$ , and then constructing the event structure

$$A \times B =_{\text{def}} \text{Pr}(\mathcal{C}(A) \times \mathcal{C}(B))$$

and its projections as  $\Pi_1 =_{\text{def}} \pi_1 \text{max}$  and  $\Pi_2 =_{\text{def}} \pi_2 \text{max}$ .

*Exercise 19.* Let  $A$  be the event structure consisting of two distinct events  $a_1 \leq a_2$  and  $B$  the event structure with a single event  $b$ . Following the method above describe the product of event structures  $A \times B$ . □

Later we shall use the following properties of  $\rightarrow$  in a product of stable families or event structures.

**Lemma 20.** *Suppose  $e \rightarrow_x e'$  in a product of stable families  $\mathcal{A} \times \mathcal{B}, \pi_1, \pi_2$ .*

- (i) *If  $e = (a, *)$  then  $e' = (a', b)$  or  $e' = (a', *)$  with  $a \rightarrow_{\pi_1 x} a'$  in  $\mathcal{A}$ .*
- (ii) *If  $e' = (a', *)$  then  $e = (a, b)$  or  $e = (a, *)$  with  $a \rightarrow_{\pi_1 x} a'$  in  $\mathcal{A}$ .*
- (iii) *If  $e = (a, b)$  and  $e' = (a', b')$  then  $a \rightarrow_{\pi_1 x} a'$  in  $\mathcal{A}$  or  $b \rightarrow_{\pi_2 x} b'$  in  $\mathcal{B}$ .*

### 3.3.2 Restriction

The *restriction* of  $\mathcal{F}$  to a subset of events  $R$  is the stable family  $\mathcal{F} \upharpoonright R =_{\text{def}} \{x \in \mathcal{F} \mid x \subseteq R\}$ . Defining  $E \upharpoonright R$ , the restriction of an event structure  $E$  to a subset of events  $R$ , to have events  $E' = \{e \in E \mid [e] \subseteq R\}$  with causal dependency and consistency induced by  $E$ , we obtain  $\mathcal{C}(E \upharpoonright R) = \mathcal{C}(E) \upharpoonright R$ .

**Proposition 21.** *Let  $\mathcal{F}$  be a stable family and  $R$  a subset of its events. Then,  $\Pr(\mathcal{F} \upharpoonright R) = \Pr(\mathcal{F}) \upharpoonright \text{max}^{-1}R$ .*

### 3.3.3 Synchronized compositions

Synchronized parallel compositions are obtained as restrictions of products to those events which are allowed to synchronize or occur asynchronously. For example, the synchronized composition of Milner’s CCS on stable families  $\mathcal{A}$  and  $\mathcal{B}$  (with labelled events) is defined as  $\mathcal{A} \times \mathcal{B} \upharpoonright R$  where  $R$  comprises events which are pairs  $(a, *)$ ,  $(*, b)$  and  $(a, b)$ , where in the latter case the events  $a$  of  $\mathcal{A}$  and  $b$  of  $\mathcal{B}$  carry complementary labels. Similarly, synchronized compositions of event structures  $A$  and  $B$  are obtained as restrictions  $A \times B \upharpoonright R$ . By Proposition 21, we can equivalently form a synchronized composition of event structures by forming the synchronized composition of their stable families of configurations, and then obtaining the resulting event structure—this has the advantage of eliminating superfluous events earlier.

### 3.3.4 Pullbacks

The construction of pullbacks can be viewed as a special case of synchronized composition. Once we have products of event structures pullbacks are obtained by restricting products to the appropriate equalizing set. Pullbacks of event structures can also be constructed via pullbacks of stable families, in a similar manner to the way we have constructed products of event structures. We obtain pullbacks of stable families as restrictions of products. Suppose  $f_1 : \mathcal{F}_1 \rightarrow \mathcal{G}$  and  $f_2 : \mathcal{F}_2 \rightarrow \mathcal{G}$  are maps of stable families. Let  $E_1, E_2$  and  $C$  be the sets of events of  $\mathcal{F}_1, \mathcal{F}_2$  and  $\mathcal{G}$ , respectively. The set  $P =_{\text{def}} \{(e_1, e_2) \mid f(e_1) = f(e_2)\}$  with projections  $\pi_1, \pi_2$  to the left and right, forms the pullback, in the category of sets, of the functions  $f_1 : E_1 \rightarrow C, f_2 : E_2 \rightarrow C$ . We obtain the pullback in stable families of  $f_1, f_2$  as the stable family  $\mathcal{P}$ , consisting of those subsets of  $P$  which are also configurations of the product  $\mathcal{F}_1 \times \mathcal{F}_2$ —its associated maps are the projections  $\pi_1, \pi_2$  from the events of  $\mathcal{P}$ .

### 3.3.5 Projection

Event structures support a simple form of hiding. Let  $(E, \leq, \text{Con})$  be an event structure. Let  $V \subseteq E$  be a subset of ‘visible’ events. Define the *projection* of  $E$  on  $V$ , to be  $E \downarrow V =_{\text{def}} (V, \leq_V, \text{Con}_V)$ , where  $v \leq_V v'$  iff  $v \leq v' \ \& \ v, v' \in V$  and  $X \in \text{Con}_V$  iff  $X \in \text{Con} \ \& \ X \subseteq V$ .

Consider a partial map of event structures  $f : E \rightarrow E'$ . Let

$$V =_{\text{def}} \{e \in E \mid f(e) \text{ is defined}\}.$$

Then  $f$  clearly factors into the composition

$$E \xrightarrow{f_0} E \downarrow V \xrightarrow{f_1} E'$$

of  $f_0$ , a partial map of event structures taking  $e \in E$  to itself if  $e \in V$  and undefined otherwise, and  $f_1$ , a total map of event structures acting like  $f$  on  $V$ .

# Chapter 4

## Games and strategies

Very general nondeterministic concurrent games and strategies are presented. The intention is to formalize distributed games in which both Player (or a team of players) and Opponent (or a team of opponents) can interact in highly distributed fashion, without, for instance, enforcing that their moves alternate. Strategies, those nondeterministic plays which compose well with copy-cat strategies, are characterized.<sup>1</sup>

### 4.1 Event structures with polarities

We shall represent both a game and a strategy in a game as an event structure with polarity, comprising an event structure together with a polarity function  $pol : E \rightarrow \{+, -\}$  ascribing a polarity  $+$  or  $-$  to its events  $E$ . The events correspond to (occurrences of) moves. The two polarities  $+/-$  express the dichotomy: Player/Opponent; Process/Environment; or Ally/Enemy. Maps of event structures with polarity are maps of event structures which preserve polarity.

### 4.2 Operations

#### 4.2.1 Dual

The *dual*,  $E^\perp$ , of an event structure with polarity  $E$  comprises a copy of the event structure  $E$  but with a reversal of polarities. It obviously extends to a functor. Write  $\bar{e} \in E^\perp$  for the event complementary to  $e \in E$  and *vice versa*.

#### 4.2.2 Simple parallel composition

This operation simply juxtaposes two event structures with polarity. Let  $(A, \leq_A, \text{Con}_A, pol_A)$  and  $(B, \leq_B, \text{Con}_B, pol_B)$  be event structures with polarity. The events of  $A \parallel B$  are  $(\{1\} \times A) \cup (\{2\} \times B)$ , their polarities unchanged, with: the only

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<sup>1</sup>This key chapter is the result of joint work with Silvain Rideau [5].

relations of causal dependency given by  $(1, a) \leq (1, a')$  iff  $a \leq_A a'$  and  $(2, b) \leq (2, b')$  iff  $b \leq_B b'$ ; a subset of events  $C$  is consistent in  $A \parallel B$  iff  $\{a \mid (1, a) \in C\} \in \text{Con}_A$  and  $\{b \mid (2, b) \in C\} \in \text{Con}_B$ . The operation extends to a functor—put the two maps in parallel. The empty event structure with polarity  $\emptyset$  is the unit w.r.t.  $\parallel$ .

### 4.3 Pre-strategies

Let  $A$  be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A *pre-strategy* in  $A$  is a total map  $\sigma : S \rightarrow A$  from an event structure with polarity  $S$ . A pre-strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey the constraints of the game; the concept will later be refined to that of *strategy* (and *winning strategy* in Section 8.1). We regard two pre-strategies  $\sigma : S \rightarrow A$  and  $\sigma' : S' \rightarrow A$  as essentially the same when they are isomorphic, and write  $\sigma \cong \sigma'$ , *i.e.* when there is an isomorphism of event structures  $\theta : S \cong S'$  such that

$$\begin{array}{ccc} S & \xrightarrow{\theta} & S' \\ & \searrow \sigma & \downarrow \sigma' \\ & & A \end{array}$$

commutes.

Let  $A$  and  $B$  be event structures with polarity. Following Joyal [6], a pre-strategy from  $A$  to  $B$  is a pre-strategy in  $A^\perp \parallel B$ , so a total map  $\sigma : S \rightarrow A^\perp \parallel B$ . It thus determines a span

$$\begin{array}{ccc} & S & \\ \sigma_1 \swarrow & & \searrow \sigma_2 \\ A^\perp & & B, \end{array}$$

of event structures with polarity where  $\sigma_1, \sigma_2$  are *partial* maps. In fact, a pre-strategy from  $A$  to  $B$  corresponds to such spans where for all  $s \in S$  either, but not both,  $\sigma_1(s)$  or  $\sigma_2(s)$  is defined. Two pre-strategies will be essentially the same when they are isomorphic as spans. Two pre-strategies  $\sigma$  and  $\tau$  from  $A$  to  $B$  are isomorphic,  $\sigma \cong \tau$ , when their spans are isomorphic, *i.e.*

$$\begin{array}{ccccc} & & S & & \\ & & | & & \\ & & \cong & & \\ & & \downarrow & & \\ & & S & & \\ \sigma_1 \swarrow & & & & \searrow \sigma_2 \\ A & & \sigma'_1 & & \sigma'_2 & B \end{array}$$

commutes. We write  $\sigma : A \dashrightarrow B$  to express that  $\sigma$  is a pre-strategy from  $A$  to  $B$ . Note a pre-strategy in a game  $A$  coincides with a pre-strategy from the empty game  $\sigma : \emptyset \dashrightarrow A$ .

### 4.3.1 Concurrent copy-cat

Identities on games are given by copy-cat strategies—strategies for Player based on copying the latest moves made by Opponent.

Let  $A$  be an event structure with polarity. The copy-cat strategy from  $A$  to  $A$  is an instance of a pre-strategy, so a total map  $\gamma_A : \mathbb{C}_A \rightarrow A^+ \parallel A$ . It describes a concurrent, or distributed, strategy based on the idea that Player moves, of +ve polarity, always copy previous corresponding moves of Opponent, of -ve polarity.

For  $c \in A^+ \parallel A$  we use  $\bar{c}$  to mean the corresponding copy of  $c$ , of opposite polarity, in the alternative component, *i.e.*

$$\overline{(1, a)} = (2, \bar{a}) \text{ and } \overline{(2, a)} = (1, \bar{a}).$$

**Proposition 22.** *Let  $A$  be an event structure with polarity. There is an event structure with polarity  $\mathbb{C}_A$  having the same events, consistency and polarity as  $A^+ \parallel A$  but with causal dependency  $\leq_{\mathbb{C}_A}$  given as the transitive closure of the relation*

$$\leq_{A^+ \parallel A} \cup \{(\bar{c}, c) \mid c \in A^+ \parallel A \text{ \& } \text{pol}_{A^+ \parallel A}(c) = +\}.$$

Moreover,

(i)  $c \rightarrow c'$  in  $\mathbb{C}_A$  iff

$$c \rightarrow c' \text{ in } A^+ \parallel A \text{ or } \text{pol}_{A^+ \parallel A}(c') = + \text{ \& } \bar{c} = c';$$

(ii)  $x \in \mathcal{C}(\mathbb{C}_A)$  iff

$$x \in \mathcal{C}(A^+ \parallel A) \text{ \& } \forall c \in x. \text{pol}_{A^+ \parallel A}(c) = + \implies \bar{c} \in x.$$

*Proof.* It can first be checked that defining

$$\begin{aligned} c \leq_{\mathbb{C}_A} c' \text{ iff } & (i) \ c \leq_{A^+ \parallel A} c' \text{ or} \\ & (ii) \ \exists c_0 \in A^+ \parallel A. \text{pol}_{A^+ \parallel A}(c_0) = + \text{ \&} \\ & \quad c \leq_{A^+ \parallel A} \bar{c}_0 \text{ \&} c_0 \leq_{A^+ \parallel A} c', \end{aligned}$$

yields a partial order. Note that

$$c \leq_{A^+ \parallel A} d \text{ iff } \bar{c} \leq_{A^+ \parallel A} \bar{d},$$

used in verifying transitivity and antisymmetry. The relation  $\leq_{\mathbb{C}_A}$  is clearly the transitive closure of  $\leq_{A^+ \parallel A}$  together with all extra causal dependencies  $(\bar{c}, c)$  where  $\text{pol}_{A^+ \parallel A}(c) = +$ . The remaining properties required for  $\mathbb{C}_A$  to be an event structure follow routinely.

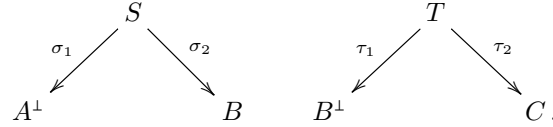
(i) From the above characterization of  $\leq_{\mathbb{C}_A}$ .

(ii) From  $\mathbb{C}_A$  and  $A^\perp \parallel A$  sharing the same consistency relation and the extra causal dependency adjoined to  $\mathbb{C}_A$ .  $\square$

Based on Proposition 22, define the *copy-cat* pre-strategy from  $A$  to  $A$  to be the pre-strategy  $\gamma_A : \mathbb{C}_A \rightarrow A^\perp \parallel A$  where  $\mathbb{C}_A$  comprises the event structure with polarity  $A^\perp \parallel A$  together with extra causal dependencies  $\bar{c} \leq_{\mathbb{C}_A} c$  for all events  $c$  with  $\text{pol}_{A^\perp \parallel A}(c) = +$ , and  $\gamma_A$  is the identity on the set of events common to both  $\mathbb{C}_A$  and  $A^\perp \parallel A$ .

### 4.3.2 Composing pre-strategies

Consider two pre-strategies  $\sigma : A \dashrightarrow B$  and  $\tau : B \dashrightarrow C$  as spans:



We show how to define their composition  $\tau \circ \sigma : A \dashrightarrow C$ . If we ignore polarities the partial maps of event structures  $\sigma_2$  and  $\tau_1$  have a common codomain, the underlying event structure of  $B$  and  $B^\perp$ . The composition  $\tau \circ \sigma$  will be constructed as a synchronized composition of  $S$  and  $T$ , in which output events of  $S$  synchronize with input events of  $T$ , followed by an operation of hiding ‘internal’ synchronization events. Only those events  $s$  from  $S$  and  $t$  from  $T$  for which  $\sigma_2(s) = \overline{\tau_1(t)}$  synchronize; note that then  $s$  and  $t$  must have opposite polarities as this is so for their images  $\sigma_2(s)$  in  $B$  and  $\tau_1(t)$  in  $B^\perp$ . The event resulting from the synchronization of  $s$  and  $t$  has indeterminate polarity and will be hidden in the composition  $\tau \circ \sigma$ .

Formally, we use the construction of synchronized composition and projection of Section 3.3.3. Via projection we hide all those events with undefined polarity.

We first define the composition of the families of configurations of  $S$  and  $T$  as a synchronized composition of stable families. We form the product of stable families  $\mathcal{C}(S) \times \mathcal{C}(T)$  with projections  $\pi_1$  and  $\pi_2$ , and then form a restriction:

$$\mathcal{C}(T) \circ \mathcal{C}(S) =_{\text{def}} \mathcal{C}(S) \times \mathcal{C}(T) \upharpoonright R$$

where

$$\begin{aligned}
 R = & \{(s, *) \mid s \in S \ \& \ \sigma_1(s) \text{ is defined}\} \cup \\
 & \{(s, t) \mid s \in S \ \& \ t \in T \ \& \ \sigma_2(s) = \overline{\tau_1(t)} \text{ with both defined}\} \cup \\
 & \{(*, t) \mid t \in T \ \& \ \tau_2(t) \text{ is defined}\}.
 \end{aligned}$$

The stable family  $\mathcal{C}(T) \circ \mathcal{C}(S)$  is the synchronized composition of the stable families  $\mathcal{C}(S)$  and  $\mathcal{C}(T)$  in which synchronizations are between events of  $S$  and  $T$  which project, under  $\sigma_2$  and  $\tau_1$  respectively, to complementary events in  $B$  and  $B^\perp$ . The stable family  $\mathcal{C}(T) \circ \mathcal{C}(S)$  represents all the configurations of the

composition of pre-strategies, including internal events arising from synchronizations. We obtain the synchronized composition as an event structure by forming  $\text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))$ , in which events are the primes of  $\mathcal{C}(T) \odot \mathcal{C}(S)$ . This synchronized composition still has internal events.

To obtain the composition of pre-strategies we hide the internal events due to synchronizations. The event structure of the composition of pre-strategies is defined to be

$$T \odot S =_{\text{def}} \text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)) \downarrow V,$$

the projection onto “visible” events,

$$V = \{p \in \text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)) \mid \exists s \in S. \max(p) = (s, *)\} \cup \\ \{p \in \text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)) \mid \exists t \in T. \max(p) = (*, t)\}.$$

Finally, the composition  $\tau \odot \sigma$  is defined by the span

$$\begin{array}{ccc} & T \odot S & \\ v_1 \swarrow & & \searrow v_2 \\ A^\perp & & C \end{array}$$

where  $v_1$  and  $v_2$  are maps of event structures, which on events  $p$  of  $T \odot S$  act so  $v_1(p) = \sigma_1(s)$  when  $\max(p) = (s, *)$  and  $v_2(p) = \tau_2(t)$  when  $\max(p) = (*, t)$ , and are undefined elsewhere.

**Proposition 23.** *Above,  $v_1$  and  $v_2$  are partial maps of event structures with polarity, which together define a pre-strategy  $v : A \rightarrow C$ . For  $x \in \mathcal{C}(T \odot S)$ ,*

$$v_1 x = \sigma_1 \pi_1 \bigcup x \text{ and } v_2 x = \tau_2 \pi_2 \bigcup x.$$

*Proof.* Consider the two maps of event structures

$$u_1 : \text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)) \xrightarrow{\Pi_1} S \xrightarrow{\sigma_1} A^\perp, \\ u_2 : \text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)) \xrightarrow{\Pi_2} T \xrightarrow{\tau_2} C,$$

where  $\Pi_1, \Pi_2$  are (restrictions of) projections of the product of event structures. *E.g.* for  $p \in \text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))$ ,  $\Pi_1(p) = s$  precisely when  $\max(p) = (s, *)$ , so  $\sigma_1(s)$  is defined, or when  $\max(p) = (s, t)$ , so  $\sigma_1(s)$  is undefined. The partial functions  $v_1$  and  $v_2$  are restrictions of the two maps  $u_1$  and  $u_2$  to the projection set  $V$ . But  $V$  consists exactly of those events in  $\text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))$  where  $u_1$  or  $u_2$  is defined. It follows that  $v_1$  and  $v_2$  are maps of event structures.

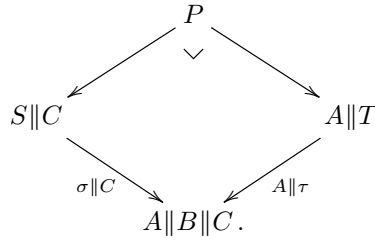
Clearly one and only one of  $v_1, v_2$  are defined on any event in  $T \odot S$  so they form a pre-strategy. Their effect on  $x \in \mathcal{C}(T \odot S)$  follows directly from their definition.  $\square$

**Proposition 24.** *Let  $\sigma : A \rightarrow B$ ,  $\tau : B \rightarrow C$  and  $v : C \rightarrow D$  be pre-strategies. The two compositions  $v \odot (\tau \odot \sigma)$  and  $(v \odot \tau) \odot \sigma$  are isomorphic.*

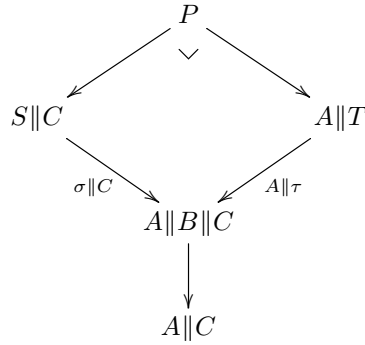
*Proof.* The natural isomorphism  $S \times (T \times U) \cong (S \times T) \times U$ , associated with the product of event structures  $S, T, U$ , restricts to the required isomorphism of spans as the synchronizations involved in successive compositions are disjoint.  $\square$

### 4.3.3 Composition via pullback

We can alternatively present the composition of pre-strategies via pullbacks.<sup>2</sup> For this section assume that the correspondence  $a \leftrightarrow \bar{a}$  between the events of  $A$  and its dual  $A^\perp$  is the identity, so  $A$  and  $A^\perp$  share the same events, though assign opposite polarities to them. Given two pre-strategies  $\sigma : S \rightarrow A^\perp \parallel B$  and  $\tau : T \rightarrow B^\perp \parallel C$ , ignoring polarities we can consider the maps on the underlying event structures, *viz.*  $\sigma : S \rightarrow A \parallel B$  and  $\tau : T \rightarrow B \parallel C$ . Viewed this way we can form the pullback in  $\mathcal{E}$  (or  $\mathcal{E}_t$ , as the maps along which we are pulling back are total)



There is an obvious partial map of event structures  $A \parallel B \parallel C \rightarrow A \parallel C$  undefined on  $B$  and acting as identity on  $A$  and  $C$ . The partial map from  $P$  to  $A \parallel C$  given by following the diagram (either way round the pullback square)



factors through the projection of  $P$  to  $V$ , those events at which the partial map is defined:

$$P \rightarrow P \downarrow V \rightarrow A \parallel C.$$

The resulting total map  $v : P \downarrow V \rightarrow A \parallel C$  gives us the composition  $\tau \circ \sigma : P \downarrow V \rightarrow A^\perp \parallel C$  once we reinstate polarities.

<sup>2</sup>I'm grateful to Nathan Bowler for the observations of this section.

### 4.3.4 Duality

A pre-strategy  $\sigma : A \multimap B$  corresponds to a dual pre-strategy  $\sigma^\perp : B^\perp \multimap A^\perp$ . This duality arises from the correspondence

$$\begin{array}{ccc} & S & \\ \sigma_1 \swarrow & & \searrow \sigma_2 \\ A^\perp & & B \end{array} \longleftrightarrow \begin{array}{ccc} & S & \\ \sigma_2 \swarrow & & \searrow \sigma_1 \\ (B^\perp)^\perp & & A^\perp \end{array}$$

It is easy to check that the dual of copy-cat,  $\gamma_{A^\perp}^\perp$ , is isomorphic, as a span, to the copy-cat of the dual,  $\gamma_{A^\perp}$ , for  $A$  an event structure with polarity. It is also straightforward, though more involved, to show that the dual of a composition of pre-strategies  $(\tau \odot \sigma)^\perp$  is isomorphic as a span to the composition  $\sigma^\perp \odot \tau^\perp$ . Duality, as usual, will save us work.

## 4.4 Strategies

This section is devoted to the main result of this chapter: that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient in order for copy-cat to behave as identity w.r.t. the composition of pre-strategies. It becomes compelling to define a (*nondeterministic*) *concurrent strategy*, in general, as a pre-strategy which is receptive and innocent.

### 4.4.1 Necessity of receptivity and innocence

The properties of *receptivity* and *innocence* of a pre-strategy, described below, will play a central role.

**Receptivity.** Say a pre-strategy  $\sigma : S \rightarrow A$  is *receptive* when  $\sigma x \xrightarrow{a} c$  &  $\text{pol}_A(a) = - \Rightarrow \exists ! s \in S. x \xrightarrow{s} c$  &  $\sigma(s) = a$ , for all  $x \in \mathcal{C}(S)$ ,  $a \in A$ . Receptivity ensures that no Opponent move which is possible is disallowed.

**Innocence.** Say a pre-strategy  $\sigma$  is *innocent* when it is both +-innocent and --innocent:

*+Innocence:* If  $s \rightarrow s'$  &  $\text{pol}(s) = +$  then  $\sigma(s) \rightarrow \sigma(s')$ .

*--Innocence:* If  $s \rightarrow s'$  &  $\text{pol}(s') = -$  then  $\sigma(s) \rightarrow \sigma(s')$ .

The definition of a pre-strategy  $\sigma : S \rightarrow A$  ensures that the moves of Player and Opponent respect the causal constraints of the game  $A$ . Innocence restricts Player further. Locally, within a configuration, Player may only introduce new relations of immediate causality of the form  $\ominus \rightarrow \oplus$ . Thus innocence gives Player the freedom to await Opponent moves before making their move, but prevents Player having any influence on the moves of Opponent beyond those stipulated in the game  $A$ ; more surprisingly, innocence also disallows any immediate causality of the form  $\oplus \rightarrow \oplus$ , purely between Player moves, not already stipulated in the game  $A$ .

Two important consequences of --innocence:

**Lemma 25.** *Let  $\sigma : S \rightarrow A$  be a pre-strategy. Suppose, for  $s, s' \in S$ , that*

$$[s] \uparrow [s'] \ \& \ \text{pol}_S(s) = \text{pol}_S(s') = - \ \& \ \sigma(s) = \sigma(s').$$

(i) *If  $\sigma$  is --innocent, then  $[s] = [s']$ .*

(ii) *If  $\sigma$  is receptive and --innocent, then  $s = s'$ .*

*[ $x \uparrow y$  expresses the compatibility of  $x, y \in \mathcal{C}(S)$ .]*

*Proof.* (i) Assume the property above holds of  $s, s' \in S$ . Assume  $\sigma$  is --innocent. Suppose  $s_1 \rightarrow s$ . Then by --innocence,  $\sigma(s_1) \rightarrow \sigma(s)$ . As  $\sigma(s') = \sigma(s)$  and  $\sigma$  is a map of event structures there is  $s_2 < s'$  such that  $\sigma(s_2) = \sigma(s_1)$ . But  $s_1, s_2$  both belong to the configuration  $[s] \cup [s']$  so  $s_1 = s_2$ , as  $\sigma$  is a map, and  $s_1 < s'$ . Symmetrically, if  $s_1 \rightarrow s'$  then  $s_1 < s$ . It follows that  $[s] = [s']$ . (ii) Now both  $[s] \xrightarrow{s} \text{c}$  and  $[s] \xrightarrow{s'} \text{c}$  with  $\sigma(s) = \sigma(s')$  where both  $s, s'$  have -ve polarity. If, further,  $\sigma$  is receptive,  $s = s'$ .  $\square$

Let  $x$  and  $x'$  be configurations of an event structure with polarity. Write  $x \subseteq^- x'$  to mean  $x \subseteq x'$  and  $\text{pol}(x' \setminus x) \subseteq \{-\}$ , i.e. the configuration  $x'$  extends the configuration  $x$  solely by events of -ve polarity. In the presence of --innocence, receptivity strengthens to the following useful property:

**Lemma 26.** *Let  $\sigma : S \rightarrow A$  be a --innocent pre-strategy. The pre-strategy  $\sigma$  is receptive iff whenever  $\sigma x \subseteq^- y$  in  $\mathcal{C}(A)$  there is a unique  $x' \in \mathcal{C}(S)$  so that  $x \subseteq x' \ \& \ \sigma x' = y$ . Diagrammatically,*

$$\begin{array}{ccc} x & \cdots \subseteq & x' \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma x & \subseteq^- & y. \end{array}$$

*[It will necessarily be the case that  $x \subseteq^- x'$ .]*

*Proof.* “if”: Clear. “Only if”: Assuming  $\sigma x \subseteq^- y$  we can form a covering chain

$$\sigma x \xrightarrow{a_1} \text{c} y_1 \cdots \xrightarrow{a_n} \text{c} y_n = y.$$

By repeated use of receptivity we obtain the existence of  $x'$  where  $x \subseteq x'$  and  $\sigma x' = y$ . To show the uniqueness of  $x'$  suppose  $x \subseteq z, z'$  and  $\sigma z = \sigma z' = y$ . Suppose that  $z \neq z'$ . Then, without loss of generality, suppose there is a  $\leq_S$ -minimal  $s' \in z'$  with  $s' \notin z$ . Then  $[s'] \subseteq z$ . Now  $\sigma(s') \in y$  so there is  $s \in z$  for which  $\sigma(s) = \sigma(s')$ . We have  $[s], [s'] \subseteq z$  so  $[s] \uparrow [s']$ . By Lemma 25(ii) we deduce  $s = s'$  so  $s' \in z$ , a contradiction. Hence,  $z = z'$ .  $\square$

It is useful to define innocence and receptivity on partial maps of event structures with polarity.

**Definition 27.** Let  $f : S \rightarrow A$  be a partial map of event structures with polarity. Say  $f$  is *receptive* when

$$f(x) \xrightarrow{a} \text{c} \ \& \ \text{pol}_A(a) = - \implies \exists! s \in S. x \xrightarrow{s} \text{c} \ \& \ f(s) = a$$

for all  $x \in \mathcal{C}(S)$ ,  $a \in A$ .

Say  $f$  is *innocent* when it is both  $+$ -innocent and  $--$ -innocent, *i.e.*

$$\begin{aligned} s \rightarrow s' \ \& \ \text{pol}(s) = + \ \& \ f(s) \text{ is defined} & \implies \\ & f(s') \text{ is defined} \ \& \ f(s) \rightarrow f(s'), \\ s \rightarrow s' \ \& \ \text{pol}(s') = - \ \& \ f(s') \text{ is defined} & \implies \\ & f(s) \text{ is defined} \ \& \ f(s) \rightarrow f(s'). \end{aligned}$$

**Proposition 28.** *A pre-strategy  $\sigma : A \multimap B$  is receptive, respectively  $+/--$ -innocent, iff both the partial maps  $\sigma_1$  and  $\sigma_2$  of its span are receptive, respectively  $+/--$ -innocent.*

**Proposition 29.** *For  $\sigma : A \multimap B$  a pre-strategy,  $\sigma_1$  is receptive, respectively  $+/--$ -innocent, iff  $(\sigma^\perp)_2$  is receptive, respectively  $+/--$ -innocent;  $\sigma$  is receptive and innocent iff  $\sigma^\perp$  is receptive and innocent.*

The next lemma will play a major role in importing receptivity and innocence to compositions of pre-strategies.

**Lemma 30.** *For pre-strategies  $\sigma : A \multimap B$  and  $\tau : B \multimap C$ , if  $\sigma_1$  is receptive, respectively  $+/--$ -innocent, then  $(\tau \circ \sigma)_1$  is receptive, respectively  $+/--$ -innocent.*

*Proof.* Abbreviate  $\tau \circ \sigma$  to  $v$ .

*Receptivity:* We show the receptivity of  $v_1$  assuming that  $\sigma_1$  is receptive. Let  $x \in \mathcal{C}(T \circ S)$  such that  $v_1 x \xrightarrow{a} c$  in  $\mathcal{C}(A^\perp)$  with  $\text{pol}_{A^\perp}(a) = -$ . By Proposition 23,  $\sigma_1 \pi_1 \cup x \xrightarrow{a} c$  with  $\pi_1 \cup x \in \mathcal{C}(S)$ . As  $\sigma_1$  is receptive there is a unique  $s \in S$  such that  $\pi_1 \cup x \xrightarrow{s} c$  in  $S$  and  $\sigma_1(s) = a$ . It follows that  $\cup x \xrightarrow{(s,*)} z$ , for some  $z$ , in  $\mathcal{C}(T) \circ \mathcal{C}(S)$ . Defining  $p =_{\text{def}} [(s, *)]_z$  we obtain  $x \xrightarrow{p} c$  and  $v_1(p) = a$ , with  $p$  the unique such event.

*Innocence:* Assume that  $\sigma_1$  is innocent. To show the  $+$ -innocence of  $v_1$  we first establish a property of the  $\rightarrow$ -relation in the event structure  $\text{Pr}(\mathcal{C}(T) \circ \mathcal{C}(S))$ , the synchronized composition of event structures  $S$  and  $T$ , before projection to  $V$ :

*If  $e \rightarrow e'$  in  $\text{Pr}(\mathcal{C}(T) \circ \mathcal{C}(S))$  with  $e \in V$ ,  $\text{pol}(e) = +$  and  $v_1(e)$  defined, then  $e' \in V$  and  $v_1(e')$  is defined.*

Assume  $e \rightarrow e'$  in  $\text{Pr}(\mathcal{C}(T) \circ \mathcal{C}(S))$ ,  $e \in V$ ,  $\text{pol}(e) = +$  and  $v_1(e)$  is defined. From the definition of  $\text{Pr}(\mathcal{C}(T) \circ \mathcal{C}(S))$ , the event  $e$  is a prime configuration of  $\mathcal{C}(T) \circ \mathcal{C}(S)$  where  $\text{max}(e)$  must have the form  $(s, *)$ , for some event  $s$  of  $S$  where  $\sigma_1(s)$  is defined. By Lemma 20,  $\text{max}(e')$  has the form  $(s', *)$  or  $(s', t)$  with  $s \rightarrow s'$  in  $S$ . Now, as  $s \rightarrow s'$  and  $\text{pol}(s) = +$ , from the  $+$ -innocence of  $\sigma_1$ , we obtain  $\sigma_1(s) \rightarrow \sigma_1(s')$  in  $A^\perp \parallel A$ . Whence  $\sigma_1(s')$  is defined ensuring  $\text{max}(e') = (s', *)$ . It follows that  $e' \in V$  and  $v_1(e')$  is defined.

Now suppose  $e \rightarrow e'$  in  $T \circ S$ . Then either

(i)  $e \rightarrow e'$  in  $\text{Pr}(\mathcal{C}(T) \circ \mathcal{C}(S))$ , or

(ii)  $e \rightarrow e_1 < e'$  in  $\text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))$  for some ‘invisible’ event  $e_1 \notin V$ .

But the above argument shows that case (ii) cannot occur when  $\text{pol}(e) = +$  and  $v_1(e)$  is defined. It follows that whenever  $e \rightarrow e'$  in  $T \odot S$  with  $\text{pol}(e) = +$  and  $v_1(e)$  defined, then  $v_1(e')$  is defined and  $v_1(e) \rightarrow v_1(e')$ , as required.

The argument showing  $--$ -innocence of  $v_1$  assuming that of  $\sigma_1$  is similar.  $\square$

**Corollary 31.** *For pre-strategies  $\sigma : A \multimap B$  and  $\tau : B \multimap C$ , if  $\tau_2$  is receptive, respectively  $+/-$ -innocent, then  $(\tau \odot \sigma)_2$  is receptive, respectively  $+/-$ -innocent.*

*Proof.* By duality using Lemma 30: if  $\tau_2$  is receptive, respectively  $+/-$ -innocent, then  $(\tau^\perp)_1$  is receptive, respectively  $+/-$ -innocent, and hence  $(\sigma^\perp \odot \tau^\perp)_1 = ((\tau \odot \sigma)^\perp)_1 = (\tau \odot \sigma)_2$  is receptive, respectively  $+/-$ -innocent.  $\square$

**Lemma 32.** *For an event structure with polarity  $A$ , the pre-strategy copy-cat  $\gamma_A : A \multimap A$  is receptive and innocent.*

*Proof. Receptive:* Suppose  $x \in \mathcal{C}(\mathbb{C}_A)$  such that  $\gamma_A x \xrightarrow{c}$  in  $\mathcal{C}(A^\perp \| A)$  where  $\text{pol}_{A^\perp \| A}(c) = -$ . Now  $\gamma_A x = x$  and  $x' =_{\text{def}} x \cup \{c\} \in \mathcal{C}(A^\perp \| A)$ . Proposition 22(ii) characterizes those configurations of  $A^\perp \| A$  which are also configurations of  $\mathbb{C}_A$ : the characterization applies to  $x$  and to its extension  $x' = x \cup \{c\}$  because of the  $-$ ve polarity of  $c$ . Hence  $x' \in \mathcal{C}(\mathbb{C}_A)$  and  $x \xrightarrow{c} x'$  in  $\mathcal{C}(\mathbb{C}_A)$ , and clearly  $c$  is unique so  $\gamma_A(c) = c$ .

*--Innocent:* Suppose  $c \rightarrow c'$  in  $\mathbb{C}_A$  and  $\text{pol}(c') = -$ . By Proposition 22(i),  $c \rightarrow c'$  in  $A^\perp \| A$ . The argument for  $+$ -innocence is similar.  $\square$

**Theorem 33.** *Let  $\sigma : A \multimap B$  be a pre-strategy from  $A$  to  $B$ . If  $\sigma \odot \gamma_A \cong \sigma$  and  $\gamma_B \odot \sigma \cong \sigma$ , then  $\sigma$  is receptive and innocent.*

*Let  $\sigma : A \multimap B$  and  $\tau : B \multimap C$  be pre-strategies which are both receptive and innocent. Then their composition  $\tau \odot \sigma : A \multimap C$  is receptive and innocent.*

*Proof.* We know the copy-cat pre-strategies  $\gamma_A$  and  $\gamma_B$  are receptive and innocent—Lemma 32. Assume  $\sigma \odot \gamma_A \cong \sigma$  and  $\gamma_B \odot \sigma \cong \sigma$ . By Lemma 30,  $(\sigma \odot \gamma_A)_1$  is receptive and innocent so  $\sigma_1$  is receptive and innocent. From its dual, Corollary 31,  $(\gamma_B \odot \sigma)_2$  so  $\sigma_2$  is receptive and innocent. Hence  $\sigma$  is receptive and innocent.

Assume that  $\sigma : A \multimap B$  and  $\tau : B \multimap C$  are receptive and innocent. The fact that  $\sigma$  is receptive and innocent ensures that  $(\tau \odot \sigma)_1$  is receptive and innocent, that  $\tau$  is receptive and innocent that  $(\tau \odot \sigma)_2$  is too. Combining, we obtain that  $\tau \odot \sigma$  is receptive and innocent.  $\square$

In other words, if a pre-strategy is to compose well with copy-cat, in the sense that copy-cat behaves as an identity w.r.t. composition, the pre-strategy must be receptive and innocent. Copy-cat behaving as identity is a hallmark of game-based semantics, so any sensible definition of concurrent strategy will have to ensure receptivity and innocence.

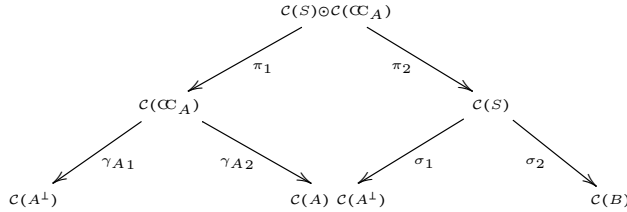
### 4.4.2 Sufficiency of receptivity and innocence

In fact, as we will now see, not only are the conditions of receptivity and innocence on pre-strategies necessary to ensure that copy-cat acts as identity. They are also sufficient.

Technically, this section establishes that for a pre-strategy  $\sigma : A \dashv\vdash B$  which is receptive and innocent both the compositions  $\sigma \circ \gamma_A$  and  $\gamma_B \circ \sigma$  are isomorphic to  $\sigma$ . We shall concentrate on the isomorphism from  $\sigma \circ \gamma_A$  to  $\sigma$ . The isomorphism from  $\gamma_B \circ \sigma$  to  $\sigma$  follows by duality.

Recall, from Section 4.3.2, the construction of the pre-strategy  $\sigma \circ \gamma_A$  as a total map  $S \circ \mathbb{C}\mathbb{C}_A \rightarrow A^\perp \parallel B$ . The event structure  $S \circ \mathbb{C}\mathbb{C}_A$  is built from the synchronized composition of stable families  $\mathcal{C}(S) \circ \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ , a restriction of the product of stable families to events

$$\begin{aligned} & \{(c, *) \mid c \in \mathbb{C}\mathbb{C}_A \ \& \ \gamma_{A_1}(c) \text{ is defined}\} \cup \\ & \{(c, s) \mid c \in \mathbb{C}\mathbb{C}_A \ \& \ s \in S \ \& \ \gamma_{A_2}(c) = \overline{\sigma_1(s)}\} \cup \\ & \{(*, s) \mid s \in S \ \& \ \sigma_2(s) \text{ is defined}\} : \end{aligned}$$



Finally  $S \circ \mathbb{C}\mathbb{C}_A$  is obtained from the prime configurations of  $\mathcal{C}(S) \circ \mathcal{C}(\mathbb{C}\mathbb{C}_A)$  whose maximum events are defined under  $\gamma_{A_1} \pi_1$  or  $\sigma_2 \pi_2$ .

We will first present the putative isomorphism from  $\sigma \circ \gamma_A$  to  $\sigma$  as a total map of event structures  $\theta : S \circ \mathbb{C}\mathbb{C}_A \rightarrow S$ . The definition of  $\theta$  depends crucially on the lemmas below. They involve special configurations of  $\mathcal{C}(S) \circ \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ , *viz.* those of the form  $\bigcup x$ , where  $x$  is a configuration of  $S \circ \mathbb{C}\mathbb{C}_A$ .

**Lemma 34.** For  $x \in \mathcal{C}(S \circ \mathbb{C}\mathbb{C}_A)$ ,

$$(c, s) \in \bigcup x \implies (\bar{c}, *) \in \bigcup x.$$

*Proof.* The case when  $pol(c) = +$  follows directly because then  $\bar{c} \rightarrow c$  in  $\mathbb{C}\mathbb{C}_A$  so  $(\bar{c}, *) \rightarrow_{\bigcup x} (c, s)$ .

Suppose the lemma fails in the case when  $pol(c) = -$ , so there is a  $\leq_{\bigcup x}$ -maximal  $(c, s) \in \bigcup x$  such that

$$pol(c) = - \ \& \ (\bar{c}, *) \notin \bigcup x. \quad (\dagger)$$

The event  $(c, s)$  cannot be maximal in  $\bigcup x$  as its maximal events take the form  $(c', *)$  or  $(*, s')$ . There must be  $e \in \bigcup x$  for which

$$(c, s) \rightarrow_{\bigcup x} e.$$

Consider the possible forms of  $e$ :

*Case  $e = (c', s')$ :* Then, by Lemma 20, either  $c \rightarrow c'$  in  $\mathbb{C}_A$  or  $s \rightarrow s'$  in  $S$ . However if  $s \rightarrow s'$  then, as  $\text{pol}(s) = +$  by innocence,  $\sigma_1(s) \rightarrow \sigma_1(s')$  in  $A^\perp$ , so  $\gamma_{A_2}(c) \rightarrow \gamma_{A_2}(c')$  in  $A$ ; but then  $c \rightarrow c'$  in  $\mathbb{C}_A$ . Either way,  $c \rightarrow c'$  in  $\mathbb{C}_A$ .

Suppose  $\text{pol}(c') = +$ . Then,

$$(c, s) \rightarrow_{\cup x} (\bar{c}, *) \rightarrow_{\cup x} (\bar{c}', *) \rightarrow_{\cup x} (c', s').$$

But this contradicts  $(c, s) \rightarrow_{\cup x} (c', s')$ .

Suppose  $\text{pol}(c') = -$ . Because  $(c, s)$  is maximal such that  $(\dagger)$ ,  $(\bar{c}', *) \in \cup x$ . But  $(\bar{c}, *) \rightarrow_{\cup x} (\bar{c}', *)$  whence  $(\bar{c}, *) \in \cup x$ , contradicting  $(\dagger)$ .

*Case  $e = (*, s')$ :* Now  $(c, s) \rightarrow_{\cup x} (*, s')$ . By Lemma 20,  $s \rightarrow s'$  in  $S$  with  $\text{pol}(s) = +$ . By innocence,  $\sigma_1(s) \rightarrow \sigma_1(s')$  and in particular  $\sigma_1(s')$  is defined, which forbids  $(*, s')$  as an event of  $\mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$ .

*Case  $e = (c', *)$ :* Now  $(c, s) \rightarrow_{\cup x} (c', *)$ . By Lemma 20,  $c \rightarrow c'$  in  $\mathbb{C}_A$ . Because  $(c, s)$  and  $(c', *)$  are events of  $\mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$  we must have  $\gamma_2(c)$  and  $\gamma_1(c')$  are defined—they are in different components of  $\mathbb{C}_A$ . By Proposition 22,  $c' = \bar{c}$ , contradicting  $(\dagger)$ .

In all cases we obtain a contradiction—hence the lemma.  $\square$

**Lemma 35.** For  $x \in \mathcal{C}(S \odot \mathbb{C}_A)$ ,

$$\sigma_1 \pi_2 \bigcup x \subseteq^- \gamma_{A_1} \pi_1 \bigcup x.$$

*Proof.* As a direct corollary of Lemma 34, we obtain:

$$\sigma_1 \pi_2 \bigcup x \subseteq \gamma_{A_1} \pi_1 \bigcup x.$$

The current lemma will follow provided all events of +ve polarity in  $\gamma_{A_1} \pi_1 \bigcup x$  are in  $\sigma_1 \pi_2 \bigcup x$ . However,  $(\bar{c}, s) \rightarrow_{\cup x} (c, *)$ , for some  $s \in S$ , when  $\text{pol}(c) = +$ .  $\square$

**Lemma 36.** For  $x \in \mathcal{C}(S \odot \mathbb{C}_A)$ ,

$$\sigma \pi_2 \bigcup x \subseteq^- \sigma \odot \gamma_A x.$$

*Proof.*

$$\begin{aligned} \sigma \pi_2 \bigcup x &= \sigma_1 \pi_2 \bigcup x \cup \sigma_2 \pi_2 \bigcup x \\ &\subseteq^- \gamma_{A_1} \pi_1 \bigcup x \cup \sigma_2 \pi_2 \bigcup x, \text{ by Lemma 35} \\ &= \sigma \odot \gamma_A x, \text{ by Proposition 23.} \end{aligned}$$

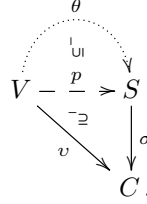
$\square$

Lemma 36 is the key to defining a map  $\theta : S \odot \mathbb{C}_A \rightarrow S$  via the following map-lifting property of receptive maps:

**Lemma 37.** Let  $\sigma : S \rightarrow C$  be a total map of event structures with polarity which is receptive and  $-$ -innocent. Let  $p : \mathcal{C}(V) \rightarrow \mathcal{C}(S)$  be a monotonic function, i.e. such that  $p(x) \subseteq p(y)$  whenever  $x \sqsubseteq y$  in  $\mathcal{C}(V)$ . Let  $v : V \rightarrow C$  be a total map of event structures with polarity such that

$$\forall x \in \mathcal{C}(V). \sigma p(x) \subseteq^- v x.$$

Then, there is a unique total map of event structures with polarity  $\theta : V \rightarrow S$  such that  $\forall x \in \mathcal{C}(V)$ .  $p(x) \sqsubseteq^- \theta x$  and  $v = \sigma\theta$  :



[We use a broken arrow to signify that  $p$  is not a map of event structures.]

*Proof.* Let  $x \in \mathcal{C}(V)$ . Then  $\sigma p(x) \sqsubseteq^- vx$ . Define  $\Theta(x)$  to be the unique configuration of  $\mathcal{C}(S)$ , determined by the receptivity of  $\sigma$ , such that

$$\begin{array}{ccc}
 p(x) & \cdots \sqsubseteq^- \cdots & \Theta(x) \\
 \sigma \downarrow & & \downarrow \sigma \\
 \sigma p(x) & \sqsubseteq^- & vx.
 \end{array}$$

Define  $\theta_x$  to be the composite bijection

$$\theta_x : x \cong vx \cong \Theta(x)$$

where the bijection  $x \cong vx$  is that determined locally by the total map of event structures  $v$ , and the bijection  $vx \cong \Theta(x)$  is the inverse of the bijection  $\sigma \upharpoonright \Theta(x) : \Theta(x) \cong vx$  determined locally by the total map  $\sigma$ .

Now, let  $y \in \mathcal{C}(V)$  with  $x \subseteq y$ . We claim that  $\theta_x$  is the restriction of  $\theta_y$ . This will follow once we have shown that  $\Theta(x) \subseteq \Theta(y)$ . Then, treating the inclusions as inclusion maps, both squares in the diagram below will commute:

$$\begin{array}{ccccc}
 \theta_y : y & \cong & vy & \cong & \Theta(y) \\
 \cup & & \cup & & \cup \\
 \theta_x : x & \cong & vx & \cong & \Theta(x)
 \end{array}$$

This will make the composite rectangle commute, *i.e.* make  $\theta_x$  the restriction of  $\theta_y$ .

To show  $\Theta(x) \subseteq \Theta(y)$  we suppose otherwise. Then there is an event  $s \in \Theta(x)$  of minimum depth w.r.t.  $\leq_S$  such that  $s \notin \Theta(y)$ . Note that  $pol(s) = -$ , as otherwise  $s \in p(x) \subseteq p(y) \subseteq \Theta(y)$ . As  $\sigma(s) \in vx \subseteq vy$  there is  $s' \in \Theta(y)$  such that  $\sigma(s') = \sigma(s)$ . From the minimality of  $s$ , both  $[s], [s'] \subseteq \Theta(y)$  ensuring the compatibility of  $[s]$  and  $[s']$ . By Lemma 25(ii),  $s = s'$  and  $s \in \Theta(y)$ —a contradiction.

By Proposition 5, the family  $\theta_x$ ,  $x \in \mathcal{C}(V)$ , determines the unique total map  $\theta : V \rightarrow S$  such that  $\theta x = \Theta(x)$ . By construction,  $p(x) \sqsubseteq^- \theta x$ , for all  $x \in \mathcal{C}(V)$ ,

and  $v = \sigma\theta$ . This property in itself ensures that  $\theta x = \Theta(x)$  so determines  $\theta$  uniquely.  $\square$

In Lemma 37, instantiate  $p: \mathcal{C}(S \odot \mathbb{C}_A) \rightarrow \mathcal{C}(S)$  to the function  $p(x) = \pi_2 \cup x$  for  $x \in \mathcal{C}(S \odot \mathbb{C}_A)$ , the map  $\sigma$  to the pre-strategy  $\sigma: S \rightarrow A^\perp \parallel B$  and  $v$  to the pre-strategy  $\sigma \odot \gamma_A$ . By Lemma 36,  $\sigma \pi_2 \cup x \sqsubseteq^- \sigma \odot \gamma_A x$ , so the conditions of Lemma 37 are met and we obtain a total map  $\theta: S \odot \mathbb{C}_A \rightarrow S$  such that  $\pi_2 \cup x \sqsubseteq^- \theta x$ , for all  $x \in \mathcal{C}(S \odot \mathbb{C}_A)$ , and  $\sigma\theta = \sigma \odot \gamma_A$ :

$$\begin{array}{ccc}
 S \odot \mathbb{C}_A & \xrightarrow{p} & S \\
 \searrow^{\sigma \odot \gamma_A} & & \downarrow \sigma \\
 & & A^\perp \parallel B
 \end{array}$$

$\overset{\text{!}_U}{\curvearrowright}$

The next lemma is used in showing  $\theta$  is an isomorphism.

**Lemma 38.** (i) Let  $z \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$ . If  $e \leq_z e'$  and  $\pi_2(e)$  and  $\pi_2(e')$  are defined, then  $\pi_2(e) \leq_S \pi_2(e')$ . (ii) The map  $\pi_2$  is surjective on configurations.

*Proof.* (i) It suffices to show when

$$e \rightarrow_z e_1 \rightarrow_z \cdots \rightarrow_z e_{n-1} \rightarrow_z e'$$

with  $\pi_2(e)$  and  $\pi_2(e')$  defined and all  $\pi_2(e_i)$ ,  $1 \leq i \leq n-1$ , undefined, that  $\pi_2(e) \leq_S \pi_2(e')$ .

*Case  $n = 1$ ,* so  $e \rightarrow_z e'$ : Use Lemma 20. If either  $e$  or  $e'$  has the form  $(*, s)$  then the other event must have the form  $(*, s')$  or  $(c', s')$  with  $s \rightarrow s'$  in  $S$ . In the remaining case  $e = (c, s)$  and  $e' = (c', s')$  with either (1)  $c \rightarrow c'$  in  $\mathbb{C}_A$ , and  $\gamma_{A_2}(c) \rightarrow \gamma_{A_2}(c')$  in  $A$ , or (2)  $s \rightarrow s'$  in  $S$ . If (1),  $\sigma_1(s) \rightarrow \sigma_1(s')$  in  $A^\perp$  where  $s, s' \in \pi_2 z$ . By Proposition 9,  $s \leq_S s'$ . In either case (1) or (2),  $\pi_2(e) \leq_S \pi_2(e')$ .

*Case  $n > 1$ :* Each  $e_i$  has the form  $(c_i, *)$ , for  $1 \leq i \leq n-1$ . By Lemma 20, events  $e$  and  $e'$  must have the form  $(c, s)$  and  $(c', s')$  with  $c \rightarrow c_1$  and  $c_{n-1} \rightarrow c'$  in  $\mathbb{C}_A$ . As  $\gamma_{A_1}(c)$  and  $\gamma_{A_2}(c_1)$  are defined,  $c_1 = \bar{c}$  and similarly  $c_{n-1} = \bar{c}'$ . Again by Lemma 20,  $c_i \rightarrow c_{i+1}$  in  $\mathbb{C}_A$  for  $1 \leq i \leq n-2$ . Consequently  $\gamma_{A_2}(c) \leq_A \gamma_{A_2}(c')$ . Now,  $s, s' \in \pi_2 z$  with  $\sigma_1(s) \leq_{A^\perp} \sigma_1(s')$ . By Proposition 9,  $s \leq_S s'$ , as required.

(ii) Let  $y \in \mathcal{C}(S)$ . Then  $\sigma_1 y \in \mathcal{C}(A^\perp)$  and by the clear surjectivity of  $\gamma_{A_2}$  on configurations there exists  $w \in \mathcal{C}(\mathbb{C}_A)$  such that  $\gamma_{A_2} w = \sigma_1 y$ . Now let

$$\begin{aligned}
 z = & \{(c, *) \mid c \in w \ \& \ \gamma_{A_1}(c) \text{ is defined}\} \\
 & \cup \{(c, s) \mid c \in w \ \& \ s \in y \ \& \ \gamma_{A_2}(c) = \sigma_1(s)\} \\
 & \cup \{(*, s) \mid s \in y \ \& \ \sigma_2(s) \text{ is defined}\}.
 \end{aligned}$$

Then, from the definition of the product of stable families—3.3.1, it can be checked that  $z \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$ . By construction,  $\pi_2 z = y$ . Hence  $\pi_2$  is surjective on configurations.  $\square$

**Theorem 39.**  $\theta: \sigma \odot \gamma_A \cong \sigma$ , an isomorphism of pre-strategies.

*Proof.* We show  $\theta$  is an isomorphism of event structures by showing  $\theta$  is rigid and both surjective and injective on configurations (Lemma 3.3 of [7]). The rest is routine.

*Rigid:* It suffices to show  $p \rightarrow p'$  in  $S \odot \mathbb{C}_A$  implies  $\theta(p) \leq_S \theta(p')$ . Suppose  $p \rightarrow p'$  in  $S \odot \mathbb{C}_A$  with  $\max(p) = e$  and  $\max(p') = e'$ . Take  $x \in \mathcal{C}(S \odot \mathbb{C}_A)$  containing  $p'$  so  $p$  too. Then

$$e \rightarrow_{\cup x} e_1 \rightarrow_{\cup x} \cdots \rightarrow_{\cup x} e_{n-1} \rightarrow_{\cup x} e'$$

where  $e, e' \in V$  and  $e_i \notin V$  for  $1 \leq i \leq n-1$ . ( $V$  consists of ‘visible’ events of the form  $(c, *)$  with  $\gamma_{A_1}(c)$  defined, or  $(*, s)$ , with  $\sigma_2(s)$  defined.)

*Case  $n = 1$ ,* so  $e \rightarrow_{\cup x} e'$ : By Lemma 20, either (i)  $e = (*, s)$  and  $e' = (*, s')$  with  $s \rightarrow s'$  in  $S$ , or (ii)  $e = (c, *)$  and  $e' = (c', *)$  with  $c \rightarrow c'$  in  $\mathbb{C}_A$ .

If (i), we observe, via  $\sigma\theta = \sigma \odot \gamma_A$ , that  $s \in \pi_2 \cup x \subseteq \theta x$  and  $\theta(p) \in \theta x$  with  $\sigma(\theta(p)) = \sigma(s)$ , so  $\theta(p) = s$  by the local injectivity of  $\sigma$ . Similarly,  $\theta(p') = s'$ , so  $\theta(p) \leq_S \theta(p')$ .

If (ii), we obtain  $\theta(p), \theta(p') \in \theta x$  with  $\sigma_1\theta(p) = \gamma_{A_1}(c)$ ,  $\sigma_1\theta(p') = \gamma_{A_1}(c')$  and  $\gamma_{A_1}(c) \rightarrow \gamma_{A_1}(c')$  in  $A^\perp$ . By Proposition 9,  $\theta(p) \leq_S \theta(p')$ .

*Case  $n > 1$ :* Note  $e_i = (c_i, s_i)$  for  $1 \leq i \leq n-1$ , and that  $s_1 \leq_S s_{n-1}$  by Lemma 38(i). Consider the case in which  $e = (c, *)$  and  $e' = (c', *)$ —the other cases are similar. By Lemma 20,  $c \rightarrow c_1$  and  $c_{n-1} \rightarrow c'$  in  $\mathbb{C}_A$ . But  $\gamma_{A_1}(c)$  and  $\gamma_{A_2}(c_1)$  are defined, so  $c_1 = \bar{c}$ , and similarly  $c_{n-1} = \bar{c}'$ . We remark that  $\theta(p) = s_1$ , by the local injectivity of  $\sigma$ , as both  $s_1 \in \pi_2 \cup x \subseteq \theta x$  and  $\theta(p) \in \theta x$  with  $\sigma(\theta(p)) = \sigma(s_1)$ . Similarly  $\theta(p') = s_{n-1}$ , whence  $\theta(p) \leq_S \theta(p')$ .

*Surjective:* Let  $y \in \mathcal{C}(S)$ . By Lemma 38(ii), there is  $z \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$  such that  $\pi_2 z = y$ . Let

$$z' = z \cup \{(c, *) \mid \text{pol}(c) = + \ \& \ \exists s \in S. (\bar{c}, s) \in z\}.$$

It is straightforward to check  $z' \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$ . Now let

$$z'' = z' \setminus \{(c, *) \mid \text{pol}(c) = - \ \& \ \forall s \in S. (\bar{c}, s) \notin z'\}.$$

Then  $z'' \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$  by the following argument. The set  $z''$  is certainly consistent, so it suffices to show

$$\text{pol}(c) = - \ \& \ (c, *) \leq_{z'} e \in z'' \implies \exists s \in S. (\bar{c}, s) \in z',$$

for all  $c \in \mathbb{C}_A$  and  $e \in z''$ . This we do by induction on the number of events between  $(c, *)$  and  $e$ . Suppose

$$\text{pol}(c) = - \ \& \ (c, *) \rightarrow_{z'} e_1 \leq_{z'} e \in z'.$$

In the case where  $e_1 = (c_1, s_1)$ , we deduce  $c \rightarrow c_1$  in  $\mathbb{C}_A$  and as  $\gamma_{A_1}(c)$  is defined while  $\gamma_{A_2}(c_1)$  is defined, we must have  $c_1 = \bar{c}$ , as required. In the case where  $e_1 = (c_1, *)$  and  $\text{pol}(c_1) = -$ , by induction, we obtain  $(\bar{c}_1, s_1) \in z'$  for some  $s_1 \in S$ . Also  $c \rightarrow c_1$ , so  $\bar{c} \rightarrow \bar{c}_1$  in  $\mathbb{C}_A$ . As  $z'$  is a configuration we must have  $(\bar{c}, s) \leq_{z'} (\bar{c}_1, s_1)$ , for some  $s \in S$ , so  $(\bar{c}, s) \in z'$ . In the case where  $e_1 = (c_1, *)$

and  $\text{pol}(c_1) = +$ , we have  $c \rightarrow c_1$  in  $\mathbb{C}_A$ . Moreover,  $(\bar{c}_1, s) \in z'$ , for some  $s \in S$ , as  $z'$  is a configuration and  $\bar{c}_1 \rightarrow c_1$  in  $\mathbb{C}_A$ . Again, from the fact that  $z'$  is a configuration, there must be  $(\bar{c}, s) \in z'$  for some  $s \in S$ . We have exhausted all cases and conclude  $z'' \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$  with  $\theta z'' = \pi_2 z = y$ , as required to show  $\theta$  is surjective on configurations.

*Injective:* Abbreviate  $\sigma \odot \gamma_A$  to  $v$ . Assume  $\theta x = \theta y$ , where  $x, y \in \mathcal{C}(S \odot \mathbb{C}_A)$ . Via the commutativity  $v = \sigma \theta$ , we observe

$$vx = \sigma \theta x = \sigma \theta y = vy.$$

Recall by Proposition 23, that  $v_1 x = \gamma_{A_1} \pi_1 \cup x = \pi_1 \cup x$ . It follows that

$$(c, *) \in \cup x \iff c \in v_1 x \iff c \in v_1 y \iff (c, *) \in \cup y.$$

Observe

$$(*, s) \in \cup x \iff \sigma_2(s) \text{ is defined \& } s \in \theta x :$$

“ $\Rightarrow$ ” by the local injectivity of  $\sigma_2$ , as  $p =_{\text{def}} [(*, s)]_{\cup x}$  yields  $\theta(p) \in \theta x$  and  $s \in \pi_2 \cup x \subseteq \theta x$  with  $\sigma_2(\theta(p)) = \sigma_2(s)$ , so  $\theta(p) = s$ ; “ $\Leftarrow$ ” as  $\sigma_2(s)$  defined and  $s \in \theta x$  entails  $s = \theta(p)$  for some  $p \in x$ , necessarily with  $\text{max}(p) = (*, s)$ . Hence

$$\begin{aligned} (*, s) \in \cup x &\iff \sigma_2(s) \text{ is defined \& } s \in \theta x \\ &\iff \sigma_2(s) \text{ is defined \& } s \in \theta y \\ &\iff (*, s) \in \cup y. \end{aligned}$$

Assuming  $(c, s) \in \cup x$  we now show  $(c, s) \in \cup y$ . (The converse holds by symmetry.) There is  $p \in x$ , such that  $(c, s) \in p$ . If  $\text{max}(p) = (*, s')$  (also in  $\cup y$  as it is visible) then as  $\pi_2$  is rigid,  $s \leq s'$  and we must have  $(c', s) \in \cup y$ . Otherwise,  $\text{max}(p) = (d, *)$  and we can suppose (by taking  $p$  minimal) that  $(c, s) \leq_{\cup x} (d', s') \rightarrow_{\cup x} (d, *)$ . But then  $\theta(p) = s' \in \theta x = \theta y$ . Also  $s \leq_S s'$ , by the rigidity of  $\pi_2$ , and, as we have seen before,  $d' = \bar{d}$  with  $d'$  -ve. Hence  $s'$  is +ve and as  $\theta y$  is a -ve extension of  $\pi_2 \cup y$  we must have  $s' \in \pi_2 \cup y$ . Hence there is  $(*, s')$  or  $(c'', s')$  in  $\cup y$ , and as  $s \leq_S s'$  there is some  $(c', s) \in \cup y$ . In both cases,  $\gamma_{A_2}(c') = \overline{\sigma_1(s)} = \gamma_{A_2}(c)$ , so  $c' = c$ , and thus  $(c, s) \in \cup y$ .

We conclude  $\cup x = \cup y$ , so  $x = y$ , as required for injectivity.  $\square$

## 4.5 Concurrent strategies

Define a *strategy* to be a pre-strategy which is receptive and innocent. We obtain a bicategory, **Games**, in which the objects are event structures with polarity—the games, the arrows from  $A$  to  $B$  are strategies  $\sigma : A \rightarrow B$  and the 2-cells are maps of spans. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies  $\odot$  (which extends to a functor on 2-cells via the functoriality of synchronized composition). The isomorphisms expressing associativity and the identity of copy-cat are those of Proposition 24 and Theorem 39 with its dual.

### 4.5.1 Alternative characterizations

#### Via saturation conditions

An alternative description of concurrent strategies exhibits the correspondence between innocence and earlier “saturation conditions,” *reflecting* specific independence, in [8, 9, 10]:

**Proposition 40.** *A strategy  $S$  in a game  $A$  comprises a total map of event structures with polarity  $\sigma : S \rightarrow A$  such that*

(i)  $\sigma x \xrightarrow{a} \text{c}$  &  $\text{pol}_A(a) = - \Rightarrow \exists! s \in S. x \xrightarrow{s} \text{c}$  &  $\sigma(s) = a$ , for all  $x \in \mathcal{C}(S)$ ,  $a \in A$ .

(ii)(+) If  $x \xrightarrow{e} \text{c}$   $x_1 \xrightarrow{e'} \text{c}$  &  $\text{pol}_S(e) = +$  in  $\mathcal{C}(S)$  and  $\sigma x \xrightarrow{\sigma(e')} \text{c}$  in  $\mathcal{C}(A)$ , then  $x \xrightarrow{e'} \text{c}$  in  $\mathcal{C}(S)$ .

(ii)(-) If  $x \xrightarrow{e} \text{c}$   $x_1 \xrightarrow{e'} \text{c}$  &  $\text{pol}_S(e') = -$  in  $\mathcal{C}(S)$  and  $\sigma x \xrightarrow{\sigma(e')} \text{c}$  in  $\mathcal{C}(A)$ , then  $x \xrightarrow{e'} \text{c}$  in  $\mathcal{C}(S)$ .

*Proof.* Note that if  $x \xrightarrow{e} \text{c}$   $x_1 \xrightarrow{e'} \text{c}$  then either  $e$  *co*  $e'$  or  $e \rightarrow e'$ . Condition (ii) is a contrapositive reformulation of innocence.  $\square$

#### Via lifting conditions

Let  $x$  and  $x'$  be configurations of an event structure with polarity. Write  $x \sqsubseteq^+ x'$  to mean  $x \subseteq x'$  and  $\text{pol}(x' \setminus x) \subseteq \{+\}$ , *i.e.* the configuration  $x'$  extends the configuration  $x$  solely by events of +ve polarity. With this notation in place we can give an attractive characterization of concurrent strategies:

**Proposition 41.** *A strategy  $S$  in a game  $A$  comprises a total map of event structures with polarity  $\sigma : S \rightarrow A$  such that*

(i) whenever  $y \sqsubseteq^+ \sigma x$  in  $\mathcal{C}(A)$  there is a (necessarily unique)  $x' \in \mathcal{C}(S)$  so that  $x' \subseteq x$  &  $\sigma x' = y$ , *i.e.*

$$\begin{array}{ccc} x' & \text{---} \sqsubseteq \text{---} & x \\ \sigma \downarrow & & \downarrow \sigma \\ y & \sqsubseteq^+ & \sigma x, \end{array}$$

and

(ii) whenever  $\sigma x \sqsubseteq^- y$  in  $\mathcal{C}(A)$  there is a unique  $x' \in \mathcal{C}(S)$  so that  $x \subseteq x'$  &  $\sigma x' = y$ , *i.e.*

$$\begin{array}{ccc} x & \text{---} \sqsubseteq \text{---} & x' \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma x & \sqsubseteq^- & y. \end{array}$$

*Proof.* Let  $\sigma : S \rightarrow A$  be a total map of event structures with polarity. It is claimed that  $\sigma$  is a strategy iff (i) and (ii).

“Only if”: Lemma 26 directly implies (ii). To establish (i) it suffices to show the seemingly weaker property (ii)’ that

$$y \overset{a}{\dashv} \sigma x \ \& \ \text{pol}(a) = + \implies \exists x' \in \mathcal{C}(S). x' \dashv x \ \& \ \sigma x' = y$$

for  $a \in A, x \in \mathcal{C}(S), y \in \mathcal{C}(A)$ . Then (ii), with  $y \sqsubseteq^+ \sigma x$ , follows by considering a covering chain  $y \dashv \dots \dashv \sigma x$ . (The uniqueness of  $x$  is a direct consequence of  $\sigma$  being a total map of event structures.) To show (ii)’, suppose  $y \overset{a}{\dashv} \sigma x$  with  $a$  +ve. Then  $\sigma(s) = a$  for some unique  $s \in x$  with  $s$  +ve. Supposing  $s$  were not  $\leq$ -maximal in  $x$ , then  $s \rightarrow s'$  for some  $s' \in x$ . By +-innocence  $a = \sigma(s) \rightarrow \sigma(s') \in \sigma x$  implying  $a$  is not  $\leq$ -maximal in  $\sigma x$ . This contradicts  $y \overset{a}{\dashv} \sigma x$ . Hence  $s$  is  $\leq$ -maximal and  $x' =_{\text{def}} x \setminus \{s\} \in \mathcal{C}(S)$  with  $x' \dashv x$  and  $\sigma x' = y$ .

“If”: Assume  $\sigma$  satisfies (i) and (ii). Clearly  $\sigma$  is receptive by (ii). We establish innocence via Proposition 40.

Suppose  $x \overset{s}{\dashv} x_1 \overset{s'}{\dashv} x'$  and  $\text{pol}(s) = +$  with  $\sigma x \overset{\sigma(s')}{\dashv} y_2$ . Then  $y_2 \overset{\sigma(s)}{\dashv} \sigma x'$  with  $\text{pol}(\sigma(s)) = +$ . From (i) we obtain a unique  $x_2 \in \mathcal{C}(S)$  such that  $x_2 \sqsubseteq x'$  and  $\sigma x_2 = y_2$ . As  $\sigma$  is a total map of event structures, we obtain  $x_2 \overset{s}{\dashv} x'$  and subsequently  $x \overset{s'}{\dashv} x_2$ , as required by Proposition 40(ii)+.

Suppose  $x \overset{s}{\dashv} x_1 \overset{s'}{\dashv} x'$  and  $\text{pol}(s') = -$  with  $\sigma x \overset{\sigma(s')}{\dashv} y_2$ . The case where  $\text{pol}(s) = +$  is covered by the previous argument: we obtain  $x \overset{s'}{\dashv} x_2$ , as required by Proposition 40(ii)-. Suppose  $\text{pol}(s) = -$ . We have

$$\sigma x \overset{\sigma(s')}{\dashv} y_2 \overset{\sigma(s)}{\dashv} \sigma x'.$$

As  $\sigma$  is already known to be receptive, we obtain

$$x \overset{e'}{\dashv} x_2 \overset{e}{\dashv} x'' \ \& \ \sigma x_2 = y_2 \ \& \ \sigma x'' = \sigma x'.$$

From the uniqueness part of (ii) we deduce  $x'' = x'$ . As  $\sigma$  is a total map of event structures,  $e = s$  and  $e' = s'$  ensuring  $x \overset{s'}{\dashv} x_2$ , as required by Proposition 40(ii)-.  $\square$

As its proof makes clear, condition (i) in Proposition 41 can be replaced by: for all  $a \in A, x \in \mathcal{C}(S), y \in \mathcal{C}(A)$ ,

$$y \overset{+}{\dashv} \sigma x \implies \exists x' \in \mathcal{C}(S). x' \dashv x \ \& \ \sigma x' = y, \quad \text{i.e.}$$

$$\begin{array}{ccc} x' & \dashv & x \\ \sigma \downarrow & & \downarrow \sigma \\ y & \overset{+}{\dashv} & \sigma x, \end{array}$$

where the relation  $\overset{+}{\dashv}$  signifies the covering relation induced by an event of +ve polarity.

**Via +-moves**

A strategy is determined by its +-moves. More precisely, a strategy  $\sigma : S \rightarrow A$  determines a monotone function  $d : \mathcal{C}(S^+) \rightarrow \mathcal{C}(A)$  given by  $d(x) = \sigma[x]_S$  for  $x \in \mathcal{C}(S^+)$ . The event structure  $S^+$  is the projection of  $S$  to its purely +-ve moves. Intuitively,  $d$  specifies the position in the game at which Player moves occur. The function  $d$  determines the original strategy  $\sigma$  via the universal property described in the proposition below.

**Proposition 42.** *Let  $\sigma : S \rightarrow A$  be a receptive --innocent pre-strategy. Define  $q : S \rightarrow S^+$  be the partial map of event structures with polarity mapping  $S$  to its projection  $S^+$  comprising only the +ve events of  $S$ , so  $qy = y^+$  for  $y \in \mathcal{C}(S)$ . Define the function  $d : \mathcal{C}(S^+) \rightarrow \mathcal{C}(A)$  to act as  $d(x) = \sigma[x]_S$  for  $x \in \mathcal{C}(S^+)$ . Then,  $d(qy) \sqsubseteq^- \sigma y$  for all  $y \in \mathcal{C}(S)$ , i.e.*

$$\begin{array}{ccc} S & \xrightarrow{q} & S^+ \\ \sigma \downarrow \text{---} \exists & \nearrow d & \\ A & & \end{array} \quad (1)$$

[The dotted line indicates that  $d$  is not a map of event structures.]

Suppose  $f : U \rightarrow A$  is a total map and  $g : U \rightarrow S^+$  a partial map of event structures with polarity such that  $d(gy) \sqsubseteq^- fy$  for all  $y \in \mathcal{C}(U)$ , i.e.

$$\begin{array}{ccc} U & \xrightarrow{g} & S^+ \\ f \downarrow \text{---} \exists & \nearrow d & \\ A & & \end{array} \quad (2)$$

Then, there is a unique total map of event structures with polarity  $\theta : U \rightarrow S$  such that  $f = \sigma\theta$  and  $g = q\theta$ ,

$$\begin{array}{ccccc} & & g & & \\ & \curvearrowright & & \curvearrowleft & \\ U & \xrightarrow{\theta} & S & \xrightarrow{q} & S^+ \\ & \searrow f & \sigma \downarrow \text{---} \exists & \nearrow d & \\ & & A & & \end{array} \quad (3)$$

*Proof.* We first check (1). Letting  $y \in \mathcal{C}(S)$ ,

$$d(qy) = d(y^+) = \sigma[y^+]_S \sqsubseteq^- y.$$

Suppose (2). Define  $p : \mathcal{C}(U) \rightarrow \mathcal{C}(S)$  by taking

$$p(z) =_{\text{def}} [gz]_S.$$

Clearly  $p$  is monotonic and

$$\sigma p(z) = \sigma[gz]_S = d(gz) \sqsubseteq^- fz$$

for all  $z \in \mathcal{C}(U)$ . By Lemma 37, there is a unique total map of event structures with polarity  $\theta: U \rightarrow S$  such that

$$f = \sigma\theta \quad \text{and} \quad \forall z \in \mathcal{C}(U). p(z) \sqsubseteq^- \theta z.$$

From the latter,  $[gz]_S \sqsubseteq^- \theta z$  from which  $gz = (gz)^+ = (\theta z)^+$ , so  $gz = q\theta z$ , for all  $z \in \mathcal{C}(U)$ . Hence we have the commuting diagram (3). Noting

$$\forall z \in \mathcal{C}(U). gz = (\theta z)^+ \iff [gz]_S \sqsubseteq^- \theta z,$$

we see that  $\theta$  is the unique map making (3) commute. □

It follows that a strategy  $\sigma$  is determined up to isomorphism by its ‘position function’  $d$  specifying at what state of the game Player moves are made. The position functions  $d$  which arise from strategies have been characterized by Alex Katovsky and GW [11].

## Chapter 5

# Deterministic strategies

This chapter concentrates on the important special case of deterministic concurrent strategies and their properties. They are shown to coincide with Mellie's and Mimram's *receptive ingenuous strategies*.

### 5.1 Definition

We say an event structure with polarity  $S$  is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} S. \text{Neg}[X] \in \text{Con}_S \implies X \in \text{Con}_S,$$

where  $\text{Neg}[X] =_{\text{def}} \{s' \in S \mid \text{pol}(s') = - \ \& \ \exists s \in X. s' \leq s\}$ . In other words,  $S$  is deterministic iff any finite set of moves is consistent when it causally depends only on a consistent set of opponent moves. Say a strategy  $\sigma : S \rightarrow A$  is deterministic if  $S$  is deterministic.

**Lemma 43.** *An event structure with polarity  $S$  is deterministic iff*

$$\forall s, s' \in S, x \in \mathcal{C}(S). \ x \xrightarrow{s} \text{c} \ \& \ x \xrightarrow{s'} \text{c} \ \& \ \text{pol}(s) = + \implies x \cup \{s, s'\} \in \mathcal{C}(S).$$

*Proof.* “*Only if*”: Assume  $S$  is deterministic,  $x \xrightarrow{s} \text{c}$ ,  $x \xrightarrow{s'} \text{c}$  and  $\text{pol}(s) = +$ . Take  $X =_{\text{def}} x \cup \{s, s'\}$ . Then  $\text{Neg}[X] \subseteq x \cup \{s\}$  so  $\text{Neg}[X] \in \text{Con}_S$ . As  $S$  is deterministic,  $X \in \text{Con}_S$  and being down-closed  $X = x \cup \{s, s'\} \in \mathcal{C}(S)$ .

“*If*”: Assume  $S$  satisfies the property stated above in the proposition. Let  $X \subseteq_{\text{fin}} S$  with  $\text{Neg}[X] \in \text{Con}_S$ . Then the down-closure  $[\text{Neg}[X]] \in \mathcal{C}(S)$ . Clearly  $[\text{Neg}[X]] \subseteq [X]$  where all events in  $[X] \setminus [\text{Neg}[X]]$  are necessarily +ve. Suppose, to obtain a contradiction, that  $X \notin \text{Con}_S$ . Then there is a maximal  $z \in \mathcal{C}(S)$  such that

$$[\text{Neg}[X]] \subseteq z \subseteq [X]$$

and some  $e \in [X] \setminus z$ , necessarily +ve, for which  $[e] \subseteq z$ . Take a covering chain

$$[e] \xrightarrow{s_1} \text{c} \ z_1 \xrightarrow{s_2} \text{c} \ \dots \ x_k \xrightarrow{s_k} \text{c} \ z_k = z.$$

As  $[e] \xrightarrow{e} [e]$  with  $e$  +ve, by repeated use of the property of the lemma—illustrated below—we obtain  $z \xrightarrow{e} z'$  in  $\mathcal{C}(S)$  with  $[Neg[X]] \subseteq z' \subseteq [X]$ , which contradicts the maximality of  $z$ .

$$\begin{array}{ccccccc} [e] & \xrightarrow{s_1} & z'_1 & \xrightarrow{s_2} & \cdots & \xrightarrow{s_k} & z'_k & = & z' \\ e \Downarrow & & e \Downarrow & & \cdots & & e \Downarrow & & \\ [e] & \xrightarrow{s_1} & z_1 & \xrightarrow{s_2} & \cdots & \xrightarrow{s_k} & z_k & = & z \end{array}$$

□

So, above, an event structure with polarity can fail to be deterministic in two ways, either with  $pol(s) = pol(s') = +$  or with  $pol(s) = +$  &  $pol(s') = -$ . In general for an event structure with polarity  $A$  the copy-cat strategy can fail to be deterministic in either way, illustrated in the examples below.

*Example 44.* (i) Take  $A$  to consist of two +ve events and one -ve event, with any two but not all three events consistent. The construction of  $\mathbb{C}_A$  is pictured:

$$\begin{array}{c} \ominus \rightarrow \oplus \\ A^\perp \ominus \rightarrow \oplus A \\ \oplus \leftarrow \ominus \end{array}$$

Here  $\gamma_A$  is not deterministic: take  $x$  to be the set of all three -ve events in  $\mathbb{C}_A$  and  $s, s'$  to be the two +ve events in the  $A$  component.

(ii) Take  $A$  to consist of two events, one +ve and one -ve event, inconsistent with each other. The construction  $\mathbb{C}_A$ :

$$\begin{array}{c} A^\perp \ominus \rightarrow \oplus A \\ \oplus \leftarrow \ominus \end{array}$$

To see  $\mathbb{C}_A$  is not deterministic, take  $x$  to be the singleton set consisting *e.g.* of the -ve event on the left and  $s, s'$  to be the +ve and -ve events on the right.

## 5.2 The bicategory of deterministic strategies

We first characterize those games for which copy-cat is deterministic; they only allow immediate conflict between events of the same polarity.

**Lemma 45.** *Let  $A$  be an event structure with polarity. The copy-cat strategy  $\gamma_A$  is deterministic iff  $A$  satisfies*

$$\forall x \in \mathcal{C}(A). x \xrightarrow{a} \& x \xrightarrow{a'} \& pol(a) = + \& pol(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A). \quad (\dagger)$$

*Proof.* “Only if”: Suppose  $x \in \mathcal{C}(A)$  with  $x \xrightarrow{a}$  and  $x \xrightarrow{a'}$  where  $pol(a) = +$  and  $pol(a') = -$ . Construct  $y =_{\text{def}} \{(1, \bar{b}) \mid b \in x\} \cup \{(1, \bar{a})\} \cup \{(2, b) \mid b \in x\}$ . Then  $y \in \mathcal{C}(\mathbb{C}_A)$  with  $y \xrightarrow{(2,a)}$  and  $y \xrightarrow{(2,a')}$ , by Proposition 22(ii). Assuming  $\mathbb{C}_A$  is

deterministic, we obtain  $y \cup \{(2, a), (2, a')\} \in \mathcal{C}(\mathbb{C}_A)$ , so  $y \cup \{(2, a), (2, a')\} \in \mathcal{C}(A^+ \| A)$ . This entails  $x \cup \{a, a'\} \in \mathcal{C}(A)$ , as required to show  $(\ddagger)$ .

“If”: Assume  $A$  satisfies  $(\ddagger)$ . It suffices to show for  $X \subseteq_{\text{fin}} \mathbb{C}_A$ , with  $X$  down-closed, that  $\text{Neg}[X] \in \text{Con}_{\mathbb{C}_A}$  implies  $X \in \text{Con}_{\mathbb{C}_A}$ . Recall  $Z \in \text{Con}_{\mathbb{C}_A}$  iff  $Z \in \text{Con}_{A^+ \| A}$ .

Let  $X \subseteq_{\text{fin}} \mathbb{C}_A$  with  $X$  down-closed. Assume  $\text{Neg}[X] \in \text{Con}_{\mathbb{C}_A}$ . Observe

- (i)  $\{c \mid c \in X \ \& \ \text{pol}(c) = -\} \subseteq \text{Neg}[X]$  and
- (ii)  $\{\bar{c} \mid c \in X \ \& \ \text{pol}(c) = +\} \subseteq \text{Neg}[X]$  as by Proposition 22,  $X$  being down-closed must contain  $\bar{c}$  if it contains  $c$  with  $\text{pol}(c) = +$ .

Consider  $X_2 =_{\text{def}} \{a \mid (2, a) \in X\}$ . Then  $X_2$  is a finite down-closed subset of  $A$ . From (i),

$$X_2^- =_{\text{def}} \{a \in X_2 \mid \text{pol}(a) = -\} \in \text{Con}_A.$$

From (ii),

$$X_2^+ =_{\text{def}} \{a \in X_2 \mid \text{pol}(a) = +\} \in \text{Con}_A.$$

We show  $(\ddagger)$  implies  $X_2 \in \text{Con}_A$ .

Define  $z^- =_{\text{def}} [X_2^-]$  and  $z^+ =_{\text{def}} [X_2^+]$ . Being down-closures of consistent sets,  $z^-, z^+ \in \mathcal{C}(A)$ . We show  $z^- \uparrow z^+$  in  $\mathcal{C}(A)$ . First note  $z^- \cap z^+ \in \mathcal{C}(A)$ . If  $a \in z^- \setminus z^- \cap z^+$  then  $\text{pol}(a) = -$ ; otherwise, if  $\text{pol}(a) = +$  then  $a \in z^+$  as well as  $a \in z^-$  making  $a \in z^- \cap z^+$ , a contradiction. Similarly, if  $a \in z^+ \setminus z^- \cap z^+$  then  $\text{pol}(a) = +$ . We can form covering chains

$$z^- \cap z^+ \xrightarrow{p_1}_{\text{c}} x_1 \xrightarrow{p_2}_{\text{c}} \dots \xrightarrow{p_k}_{\text{c}} x_k = z^- \quad \text{and} \quad z^- \cap z^+ \xrightarrow{n_1}_{\text{c}} y_1 \xrightarrow{n_2}_{\text{c}} \dots \xrightarrow{n_l}_{\text{c}} y_l = z^+$$

where each  $p_i$  is +ve and each  $n_j$  is -ve.

Consequently, by repeated use of  $(\ddagger)$ , we obtain  $x_k \cup y_l \in \mathcal{C}(A)$ , *i.e.*  $z^+ \cup z^- \in \mathcal{C}(A)$ , as is illustrated below. But  $X_2 \subseteq z^+ \cup z^-$ , so  $X_2 \in \text{Con}_A$ . A similar argument shows  $X_1 =_{\text{def}} \{a \in A^+ \mid (1, a) \in X\} \in \text{Con}_{A^+}$ . It follows that  $X \in \text{Con}_{A^+ \| A}$ , so  $X \in \text{Con}_{\mathbb{C}_A}$  as required.

$$\begin{array}{cccccccc}
y_l & \xrightarrow{p_1}_{\text{c}} & x_1 \cup y_l & \xrightarrow{p_2}_{\text{c}} & x_2 \cup y_l & \xrightarrow{p_3}_{\text{c}} & \dots & \xrightarrow{p_k}_{\text{c}} & x_k \cup y_l \\
n_l \updownarrow & & n_l \updownarrow & & n_l \updownarrow & & & & n_l \updownarrow \\
\vdots & & \vdots & & \vdots & \dots & \dots & \dots & \vdots \\
n_2 \updownarrow & & n_2 \updownarrow & & n_2 \updownarrow & & & & n_2 \updownarrow \\
y_1 & \xrightarrow{p_1}_{\text{c}} & x_1 \cup y_1 & \xrightarrow{p_2}_{\text{c}} & x_2 \cup y_1 & \xrightarrow{p_3}_{\text{c}} & \dots & \xrightarrow{p_k}_{\text{c}} & x_k \cup y_1 \\
n_1 \updownarrow & & n_1 \updownarrow & & n_1 \updownarrow & & & & n_1 \updownarrow \\
z^- \cap z^+ & \xrightarrow{p_1}_{\text{c}} & x_1 & \xrightarrow{p_2}_{\text{c}} & x_2 & \xrightarrow{p_3}_{\text{c}} & \dots & \xrightarrow{p_k}_{\text{c}} & x_k
\end{array}$$

□

Via the next lemma, when games satisfy  $(\ddagger)$  we can simplify the condition for a strategy to be deterministic.

**Lemma 46.** *Let  $\sigma : S \rightarrow A$  be a strategy. Suppose  $x \xrightarrow{s} y$  &  $x \xrightarrow{s'} y'$  &  $\text{pol}_S(s) = -$ . Then,  $\sigma y \uparrow \sigma y'$  in  $\mathcal{C}(A) \implies y \uparrow y'$  in  $\mathcal{C}(S)$ .*

*Proof.* Assume  $\sigma y \uparrow \sigma y'$  in  $\mathcal{C}(A)$ , so  $\sigma y' \xrightarrow{\sigma(s)} \sigma y \cup \sigma y'$  in  $\mathcal{C}(A)$ . As  $\sigma(s)$  is  $-ve$ , by receptivity, there is a unique  $s'' \in S$ , necessarily  $-ve$ , such that  $\sigma(s'') = \sigma(s)$  and  $y' \xrightarrow{s''} x \cup \{s', s''\}$  in  $\mathcal{C}(S)$ . In particular,  $x \cup \{s', s''\} \in \mathcal{C}(S)$ . By  $--$ -innocence, we cannot have  $s' \rightarrow s''$ , so  $x \cup \{s''\} \in \mathcal{C}(S)$ . But now  $x \xrightarrow{s}$  and  $x \xrightarrow{s''}$  with  $\sigma(s) = \sigma(s'')$  and both  $s, s''$   $-ve$  and hence  $s'' = s$  by the uniqueness part of receptivity. We conclude that  $x \cup \{s', s\} \in \mathcal{C}(S)$  so  $y \uparrow y'$ .  $\square$

**Corollary 47.** *Assume  $A$  satisfies  $(\ddagger)$  of Lemma 45. A strategy  $\sigma : S \rightarrow A$  is deterministic iff it is weakly-deterministic, i.e. for all  $+ve$  events  $s, s' \in S$  and configurations  $x \in \mathcal{C}(S)$ ,*

$$x \xrightarrow{s} \text{ & } x \xrightarrow{s'} \implies x \cup \{s, s'\} \in \mathcal{C}(S).$$

*Proof.* “Only if”: clear. “If”: Let  $x \xrightarrow{s}$  and  $x \xrightarrow{s'}$  where  $\text{pol}_S(s) = +$ . For  $S$  to be deterministic we require  $x \cup \{s, s'\} \in \mathcal{C}(S)$ . The above assumption ensures this when  $\text{pol}_S(s') = +$ . Otherwise  $\text{pol}_S(s') = -$  with  $\sigma x \xrightarrow{\sigma(s)}$  and  $\sigma x \xrightarrow{\sigma(s')}$ . As  $A$  satisfies  $(\ddagger)$ ,  $\sigma x \cup \sigma(s), \sigma(s') \in \mathcal{C}(A)$ . Now by Lemma 46,  $x \cup \{s, s'\} \in \mathcal{C}(S)$ .  $\square$

**Lemma 48.** *The composition  $\tau \circ \sigma$  of deterministic strategies  $\sigma$  and  $\tau$  is deterministic.*

*Proof.* Let  $\sigma : S \rightarrow A^1 \parallel B$  and  $\tau : T \rightarrow B^1 \parallel C$  be deterministic strategies. The composition  $T \circ S$  is constructed as  $\text{Pr}(\mathcal{C}(T) \circ \mathcal{C}(S)) \downarrow V$ , a synchronized composition of event structures  $S$  and  $T$  projected to visible events  $e \in V$  where  $\text{max}(e)$  has the form  $(s, *)$  or  $(*, t)$ .

We first note a fact about the effect of internal, or “invisible,” events not in  $V$  on configurations of  $\mathcal{C}(T) \circ \mathcal{C}(S)$ . If

$$z \xrightarrow{(s,t)} w \text{ & } z \xrightarrow{(s',t')} w' \text{ & } w \uparrow w' \tag{1}$$

within  $\mathcal{C}(T) \circ \mathcal{C}(S)$ , then either

$$\pi_1 z \xrightarrow{s} \pi_1 w \text{ & } \pi_1 z \xrightarrow{s'} \pi_1 w' \text{ & } \pi_1 w \uparrow \pi_1 w', \tag{2}$$

within  $\mathcal{C}(S)$ , or

$$\pi_2 z \xrightarrow{t} \pi_2 w \text{ & } \pi_2 z \xrightarrow{t'} \pi_2 w' \text{ & } \pi_2 w \uparrow \pi_2 w', \tag{3}$$

within  $\mathcal{C}(T)$ . Assume (1). If  $t = t'$  then  $\sigma(s) = \overline{\tau(t)} = \overline{\tau(t')} = \sigma(s')$  and we obtain (2) as  $\sigma$  is a map of event structures. Similarly if  $s = s'$  then (3). Supposing  $s \neq s'$  and  $t \neq t'$  then if both (2) and (3) failed we could construct a configuration

$z' =_{\text{def}} z \cup \{(s, t), (s', t)\}$  of  $\mathcal{C}(T) \odot \mathcal{C}(S)$ , contradicting (1); it is easy to check that  $z'$  is a configuration of the product  $\mathcal{C}(S) \times \mathcal{C}(T)$  and its events are clearly within the restriction used in defining the synchronized composition.

We now show the impossibility of (2) and (3), and so (1). Assume (2) (case (3) is similar). One of  $s$  or  $s'$  being +ve would contradict  $S$  being deterministic. Suppose otherwise, that both  $s$  and  $s'$  are -ve. Then, because  $\sigma$  is a strategy, by Lemma 46, we have

$$\sigma_2 \pi_1 w \uparrow \sigma_2 \pi_1 w'$$

in  $\mathcal{C}(B)$ . Also, then both  $t$  and  $t'$  are +ve ensuring  $\pi_2 w \uparrow \pi_2 w'$  in  $\mathcal{C}(T)$ , as  $T$  is deterministic. This entails

$$\tau_1 \pi_2 w \uparrow \tau_1 \pi_2 w'$$

in  $\mathcal{C}(B^\perp)$ . But  $\sigma_2 \pi_1 w$  and  $\tau_1 \pi_2 w$ , respectively  $\sigma_2 \pi_1 w'$  and  $\tau_1 \pi_2 w'$ , are the same configurations on the common event structure underlying  $B$  and  $B^\perp$ , of which we have obtained contradictory statements of compatibility.

As (1) is impossible, it follows that

$$z \xrightarrow{(s,t)} w \ \& \ z \xrightarrow{(s',t')} w' \implies w \uparrow w' \quad (4)$$

within  $\mathcal{C}(T) \odot \mathcal{C}(S)$ .

Finally, we can show that  $\tau \odot \sigma$  is deterministic. Suppose  $x \xrightarrow{p} y$  and  $x \xrightarrow{p'} y'$  in  $\mathcal{C}(T \odot S)$  with  $\text{pol}(p) = +$ . Then,

$$\bigcup x \xrightarrow{e_1} z_1 \xrightarrow{e_2} \dots \xrightarrow{e_k} z_k = \bigcup y \quad \text{and} \quad \bigcup x \xrightarrow{e'_1} z'_1 \xrightarrow{e'_2} \dots \xrightarrow{e'_l} z'_l = \bigcup y'$$

in  $\mathcal{C}(T) \odot \mathcal{C}(S)$ , where  $e_k = \max(p)$  and  $e'_l = \max(p')$ , and the events  $e_i$  and  $e'_j$  otherwise have the form  $e_i = (s_i, t_i)$ , when  $1 \leq i < k$ , and  $e'_j = (s'_j, t'_j)$ , when  $1 \leq j < l$ . By repeated use of (4) we obtain  $z_{k-1} \uparrow z'_{l-1}$ . (The argument is like that ending the proof of Lemma 45, though with the minor difference that now we may have  $e_i = e'_j$ .) We obtain  $w =_{\text{def}} z_{k-1} \cup z'_{l-1} \in \mathcal{C}(T) \odot \mathcal{C}(S)$  with  $w \xrightarrow{e_k}$  and  $w \xrightarrow{e'_l}$  and  $\text{pol}(e_k) = +$ .

Now,  $w \cup \{e_k, e'_l\} \in \mathcal{C}(T) \odot \mathcal{C}(S)$  provided  $w \cup \{e_k, e'_l\} \in \mathcal{C}(S) \times \mathcal{C}(T)$ . Inspect the definition of configurations of the product of stable families in Section 3.3.1. If  $e_k$  and  $e'_l$  have the form  $(s, *)$  and  $(s', *)$  respectively, then determinacy of  $S$  ensures that the projection  $\pi_1 w \cup \{s, s'\} \in \mathcal{C}(S)$  whence  $w \cup \{e_k, e'_l\}$  meets the conditions needed to be in  $\mathcal{C}(S) \times \mathcal{C}(T)$ . Similarly,  $w \cup \{e_k, e'_l\} \in \mathcal{C}(S) \times \mathcal{C}(T)$  if  $e_k$  and  $e'_l$  have the form  $(*, t)$  and  $(*, t')$ . Otherwise one of  $e_k$  and  $e'_l$  has the form  $(s, *)$  and the other  $(*, t)$ . In this case again an inspection of the definition of configurations of the product yields  $w \cup \{e_k, e'_l\} \in \mathcal{C}(S) \times \mathcal{C}(T)$ . Forming the set of primes of  $w \cup \{e_k, e'_l\}$  in  $V$  we obtain  $x \cup \{p, p'\} \in \mathcal{C}(T \odot S)$ .

This establishes that  $T \odot S$  is deterministic.  $\square$

We thus obtain a sub-bicategory **DGames** of **Games**; its objects satisfy  $(\ddagger)$  of Lemma 45 and its maps are deterministic strategies.

### 5.3 A category of deterministic strategies

In fact, **DGames** is equivalent to an order-enriched category via the following lemma. It says weakly-deterministic strategies in a game  $A$  are essentially certain subfamilies of configurations  $\mathcal{C}(A)$ , for which we give a characterization in the case of deterministic strategies. Recall, from Corollary 47, a weakly-deterministic strategy  $\sigma : S \rightarrow A$  is a strategy in which for all +ve events  $s, s' \in S$  and configurations  $x \in \mathcal{C}(S)$ ,

$$x \xrightarrow{s} \text{c} \ \& \ x \xrightarrow{s'} \text{c} \implies x \cup \{s, s'\} \in \mathcal{C}(S).$$

**Lemma 49.** *Let  $\sigma : S \rightarrow A$  be a weakly-deterministic strategy. Then,*

$$\sigma x \subseteq \sigma y \implies x \subseteq y$$

for all  $x, y \in \mathcal{C}(S)$ . In particular, a weakly-deterministic strategy  $\sigma$  is injective on configurations, i.e.,  $\sigma x = \sigma y$  implies  $x = y$ , for all  $x, y \in \mathcal{C}(S)$  (so is mono as a map of event structures).

*Proof.* Let  $\sigma : S \rightarrow A$  be a weakly-deterministic strategy. We show

$$x \supseteq z \text{-c} y \ \& \ \sigma y \subseteq \sigma x \implies y \subseteq x,$$

for  $x, y, z \in \mathcal{C}(S)$ , by induction on  $|x \setminus z|$ .

Suppose  $x \supseteq z \xrightarrow{e} \text{c} y$  and  $\sigma y \subseteq \sigma x$ . There are  $x_1$  and event  $e_1 \in S$  such that  $z \xrightarrow{e_1} \text{c} x_1 \subseteq x$ . If  $\sigma(e_1) = \sigma(e)$  then  $e_1$  and  $e$  have the same polarity; if -ve,  $e_1 = e$  by receptivity; if +ve,  $e_1 = e$  because  $\sigma$  is weakly-deterministic, using its local injectivity. Either way  $y \subseteq x$ . Suppose  $\sigma(e_1) \neq \sigma(e)$ . We show in all cases  $y \cup \{e_1\} \subseteq x$ , so  $y \subseteq x$ .

*Case  $\text{pol}(e_1) = \text{pol}(e) = +$ :* As  $\sigma$  is weakly-deterministic,  $e_1$  and  $e$  are concurrent giving  $x_1 \xrightarrow{e} \text{c} y \cup \{e_1\}$ . By induction we obtain  $y \cup \{e_1\} \subseteq x$ .

*Case  $\text{pol}(e) = -$  or  $\text{pol}(e_1) = -$ :* From Lemma 46, we deduce that  $e_1$  and  $e$  are concurrent yielding  $x_1 \xrightarrow{e} \text{c} y \cup \{e_1\}$ , and by induction  $y \cup \{e_1\} \subseteq x$ .

Another, simpler induction on  $|y \setminus z|$  now yields

$$x \supseteq z \subseteq y \ \& \ \sigma y \subseteq \sigma x \implies y \subseteq x,$$

for  $x, y, z \in \mathcal{C}(S)$ , from which the result follows (taking  $z$  to be, for instance,  $\emptyset$  or  $x \cap y$ ). Injectivity of  $\sigma$  as a function on configurations is now obvious.  $\square$

A deterministic strategy  $\sigma : S \rightarrow A$  determines, as the image of the configurations  $\mathcal{C}(S)$ , a subfamily  $F =_{\text{def}} \sigma \mathcal{C}(S)$  of configurations of  $\mathcal{C}(A)$ , satisfying:

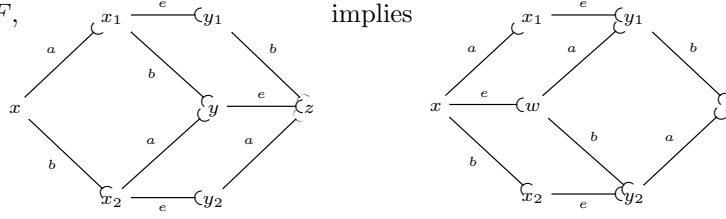
*reachability:*  $\emptyset \in F$  and if  $x \in F$  there is a covering chain  $\emptyset \xrightarrow{a_1} \text{c} x_1 \xrightarrow{a_2} \text{c} \dots \xrightarrow{a_k} \text{c} x_k = x$  within  $F$ ;

*determinacy:* If  $x \xrightarrow{a} \text{c}$  and  $x \xrightarrow{a'} \text{c}$  in  $F$  with  $\text{pol}_A(a) = +$ , then  $x \cup \{a, a'\} \in F$ ;

*receptivity:* If  $x \in F$  and  $x \xrightarrow{a} \text{c}$  in  $\mathcal{C}(A)$  and  $\text{pol}_A(a) = -$ , then  $x \cup \{a\} \in F$ ;

*+innocence:* If  $x \xrightarrow{a} \text{c} x_1 \xrightarrow{a'} \text{c}$  &  $\text{pol}_A(a) = +$  in  $F$  and  $x \xrightarrow{a'} \text{c}$  in  $\mathcal{C}(A)$ , then  $x \xrightarrow{a'} \text{c}$  in  $F$  (here receptivity implies --innocence);

*cube:* In  $F$ ,



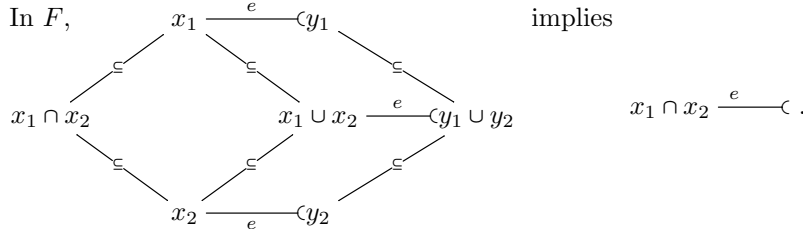
**Theorem 50.** A subfamily  $F \subseteq \mathcal{C}(A)$  satisfies the axioms above iff there is a deterministic strategy  $\sigma : S \rightarrow A$  such that  $F = \sigma\mathcal{C}(S)$ , the image of  $\mathcal{C}(S)$  under  $\sigma$ .

*Proof.* (Sketch) It is routine to check that  $F$ , the image  $\sigma\mathcal{C}(S)$  of a deterministic strategy, satisfies the axioms. Conversely, suppose a subfamily  $F \subseteq \mathcal{C}(A)$  satisfies the axioms. We show  $F$  is a stable family. First note that from the axioms of determinacy and receptivity we can deduce:

$$\text{if } x \xrightarrow{a} c \text{ and } x \xrightarrow{a'} c \text{ in } F \text{ with } x \cup \{a, a'\} \in \mathcal{C}(A), \text{ then } x \cup \{a, a'\} \in F.$$

By repeated use of this property, using their reachability, if  $x, y \in F$  and  $x \uparrow y$  in  $\mathcal{C}(A)$  then  $x \cup y \in F$ ; the proof also yields a covering chain from  $x$  to  $x \cup y$  and from  $y$  to  $x \cup y$ . (In particular, if  $x \subseteq y$  in  $F$ , then there is a covering chain from  $x$  to  $y$ —a fact we shall use shortly.) Thus, if  $x \uparrow y$  in  $F$  then  $x \cup y \in F$ . As also  $\emptyset \in F$ , we obtain Completeness, required of a stable family. Coincidence-freeness is a direct consequence of reachability. Repeated use of the cube axiom yields

*Cube:* In  $F$ ,



We use *Cube* to show stability. Assume  $v \uparrow w$  in  $F$ . Let  $z \in F$  be maximal such that  $z \subseteq v, w$ . We show  $z = v \cap w$ . Suppose not. Then, forming covering chains in  $F$ ,

$$z \xrightarrow{c_1} c v_1 \xrightarrow{c_2} c \dots \xrightarrow{c_k} c v_k = v \quad \text{and} \quad z \xrightarrow{d_1} c w_1 \xrightarrow{d_2} c \dots \xrightarrow{d_l} c w_l = w,$$

there are  $c_i$  and  $d_j$  such that  $c_i = d_j$ , where we may assume  $c_i$  is the earliest event to be repeated as some  $d_j$ . Write  $e =_{\text{def}} c_i = d_j$ . Now,  $v_{i-1} \cap w_{j-1} = z$ . Also, being bounded above  $v_{i-1} \cup w_{j-1} \in F$  and  $v_i \cup w_j \in F$ . We have an instance of *Cube*: take  $x_1 = v_{i-1}$ ,  $x_2 = w_{j-1}$ ,  $y_1 = v_i$  and  $y_2 = w_j$ . Hence  $z \xrightarrow{e} c$  and  $z \cup \{e\} \subseteq x, y$ —contradicting the maximality of  $z$ . Therefore  $z = v \cap w$ , as required for stability.

Now we can form an event structure  $S =_{\text{def}} \text{Pr}(F)$ . The inclusion  $F \subseteq \mathcal{C}(A)$  induces a total map  $\sigma : S \rightarrow A$  for which  $F = \sigma\mathcal{C}(S)$ . Note that  $--$ -innocence (*viz.*

if  $x \xrightarrow{a} c$   $x_1 \xrightarrow{a'} c$  &  $pol_A(a') = -$  in  $F$  and  $x \xrightarrow{a'} c$  in  $\mathcal{C}(A)$ , then  $x \xrightarrow{a'} c$  in  $F$  is a direct consequence of receptivity. That  $S$  is deterministic follows from determinacy, that  $\sigma$  is a strategy from the axioms of receptivity and +-innocence.  $\square$

We can thus identify deterministic strategies from  $A$  to  $B$  with subfamilies of  $\mathcal{C}(A^+ \parallel B)$  satisfying the axioms above. Through this identification we obtain an order-enriched category of deterministic strategies (presented as subfamilies) equivalent to **DGames**.

# Chapter 6

## Games people play

We briefly and incompletely examine special cases of nondeterministic concurrent games in the literature.

### 6.1 Categories for games

We remark that event structures with polarity appear to provide a rich environment in which to explore structural properties of games and strategies. There are adjunctions

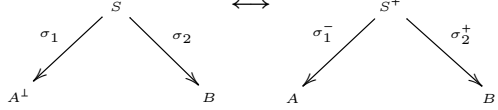
$$\begin{array}{ccccc}
 \mathcal{P}\mathcal{A}_r & \xleftarrow{\tau} & \mathcal{P}\mathcal{F}_r & \xleftarrow{\tau} & \mathcal{P}\mathcal{E}_r & \xleftarrow{\tau} & \mathcal{P}\mathcal{E}_t \\
 \downarrow \dashv & \uparrow & \downarrow \dashv & \uparrow & & & \\
 \mathcal{P}\mathcal{A}_r^\# & \xleftarrow{\tau} & \mathcal{P}\mathcal{F}_r^\# & & & & 
 \end{array}$$

relating  $\mathcal{P}\mathcal{E}_t$ , the category of event structures with polarity with total maps, to subcategories  $\mathcal{P}\mathcal{E}_r$ , with rigid maps,  $\mathcal{P}\mathcal{F}_r$  of forest-like (or filiform) event structures with rigid maps, and  $\mathcal{P}\mathcal{A}_r$ , its full subcategory where polarities alternate along a branch; in  $\mathcal{P}\mathcal{F}_r^\#$  and  $\mathcal{P}\mathcal{A}_r^\#$  distinct branches are inconsistent. We shall mainly be considering games in  $\mathcal{P}\mathcal{E}_t$ . Lamarche games and those of sequential algorithms belong to  $\mathcal{P}\mathcal{A}_r$  [12]. Conway games inhabit  $\mathcal{P}\mathcal{F}_r^\#$ , in fact a coreflective subcategory of  $\mathcal{P}\mathcal{E}_t$  as the inclusion is now full; Conway’s ‘sum’ is obtained by applying the right adjoint to the  $\parallel$ -composition of Conway games in  $\mathcal{P}\mathcal{E}_t$ . Further refinements are possible. The ‘simple games’ of [13, 14] belong to  $\mathcal{P}\mathcal{A}_r^{-\#}$ , the coreflective subcategory of  $\mathcal{P}\mathcal{A}_r^\#$  comprising “polarized” games, starting with moves of Opponent. The ‘tensor’ of simple games is recovered by applying the right adjoint of  $\mathcal{P}\mathcal{A}_r^{-\#} \hookrightarrow \mathcal{P}\mathcal{E}_t$  to their  $\parallel$ -composition in  $\mathcal{P}\mathcal{E}_t$ . Generally, the right adjoints, got by composition, from  $\mathcal{P}\mathcal{E}_t$  to the other categories fail to conserve immediate causal dependency. Such facts led Melliès *et al.* to the insight that uses of pointers in game semantics can be an artifact of working with models of games which do not take account of the independence of moves [15, 10].

## 6.2 Related work—early results

### 6.2.1 Stable spans, profunctors and stable functions

The sub-bicategory of **Games** where the events of games are purely +ve is equivalent to the bicategory of stable spans [7]. In this case, strategies correspond to *stable spans*:



where  $S^+$  is the projection of  $S$  to its +ve events;  $\sigma_2^+$  is the restriction of  $\sigma_2$  to  $S^+$ , necessarily a rigid map by innocence;  $\sigma_1^-$  is a *demand map* taking  $x \in \mathcal{C}(S^+)$  to  $\sigma_1^-(x) = \sigma_1[x]$ ; here  $[x]$  is the down-closure of  $x$  in  $S$ . Composition of stable spans coincides with composition of their associated profunctors—see [3]. If we further restrict strategies to be deterministic (and, strictly, event structures to be countable) we obtain a bicategory equivalent to Berry’s *dI-domains and stable functions*.

### 6.2.2 Ingenuous strategies

Via Theorem 50, deterministic concurrent strategies coincide with the *receptive ingenuous strategies* of Mellès and Mimram [10].

### 6.2.3 Closure operators

In [16], deterministic strategies are presented as closure operators. A deterministic strategy  $\sigma : S \rightarrow A$  determines a closure operator  $\varphi$  on possibly infinite configurations  $\mathcal{C}^\infty(S)$ : for  $x \in \mathcal{C}^\infty(S)$ ,

$$\varphi(x) = x \cup \{s \in S \mid \text{pol}(s) = + \ \& \ \text{Neg}[\{s\}] \subseteq x\}.$$

Clearly  $\varphi$  preserves intersections of configurations and is continuous. The closure operator  $\varphi$  on  $\mathcal{C}^\infty(S)$  induces a *partial* closure operator  $\varphi_p$  on  $\mathcal{C}^\infty(A)$ . This in turn determines a closure operator  $\varphi_p^\top$  on  $\mathcal{C}^\infty(A)^\top$ , where configurations are extended with a top  $\top$ , *cf.* [16]: take  $y \in \mathcal{C}^\infty(A)^\top$  to the least, fixed point of  $\varphi_p$  above  $y$ , if such exists, and  $\top$  otherwise.

### 6.2.4 Simple games

“*Simple games*” [13, 14] arise when we restrict **Games** to objects and deterministic strategies in  $\mathcal{PA}_r^{\#}$ , described in Section 6.1.

### 6.2.5 Extensions

Games, such as those of [17, 18], allowing copying are being systematized through the use of monads and comonads [14], work now feasible on event structures with

symmetry [7]. Nondeterministic strategies can potentially support probability as probabilistic or stochastic event structures [19] to become probabilistic or stochastic strategies.



# Chapter 7

## Strategies as profunctors

This chapter relates strategies to profunctors, a generalization of relations from sets to categories, and composition on strategies to composition of profunctors. Profunctors themselves provide a rich framework in which to generalize domain theory in a way that is arguably closer to that initiated by Dana Scott than game semantics [20, 21]. Early connections are made with bistructures.

### 7.1 The Scott order in games

Let  $A$  be an event structure with polarity. The  $\sqsubseteq$ -order on its finite configurations is obtained as compositions of two more fundamental orders  $(\sqsubseteq^+ \cup \sqsubseteq^-)^+$ . For  $x, y \in \mathcal{C}^\infty(A)$ ,

$$\begin{aligned} x \sqsubseteq^- y &\text{ iff } x \sqsubseteq y \ \& \ \text{pol}_A(y \setminus x) \sqsubseteq \{-\}, \text{ and} \\ x \sqsubseteq^+ y &\text{ iff } x \sqsubseteq y \ \& \ \text{pol}_A(y \setminus x) \sqsubseteq \{+\}. \end{aligned}$$

We use  $\supseteq$  as the converse order to  $\sqsubseteq$ . Define a new order, the *Scott order*, between configurations  $x, y \in \mathcal{C}^\infty(A)$ , by

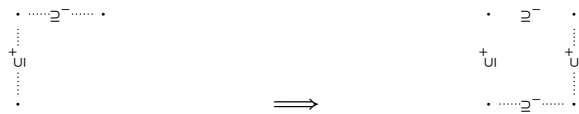
$$x \sqsubseteq_A y \iff \exists z \in \mathcal{C}^\infty(A). x \supseteq^- z \sqsubseteq^+ y.$$

It is an easy exercise to show that when such a  $z$  exists it is necessarily  $x \cap y$ .

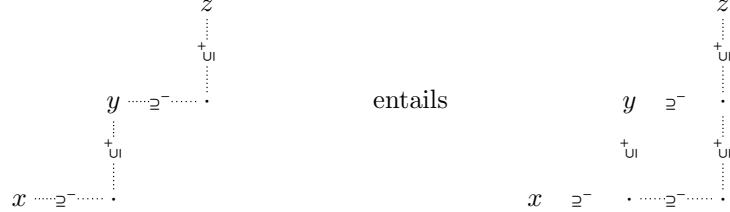
**Proposition 51.** *Let  $A$  be an event structure with polarity.*

- (i) *If  $x \sqsubseteq^+ w \supseteq^- y$  in  $\mathcal{C}^\infty(A)$ , then  $x \supseteq^- x \cap y \sqsubseteq^+ y$  in  $\mathcal{C}^\infty(A)$ .*
- (ii)  *$(\mathcal{C}^\infty(A), \sqsubseteq_A)$  is a partial order.*

*Proof.* (i) Assume  $x \sqsubseteq^+ w \supseteq^- y$  in  $\mathcal{C}^\infty(A)$ . Clearly  $x \supseteq x \cap y$ . Suppose  $a \in x$  and  $\text{pol}_A(a) = +$ . Then  $a \in w$ , and because only  $-$ ve events are lost from  $w$  in  $w \supseteq^- y$  we obtain  $a \in y$ , so  $a \in x \cap y$ . It follows that  $x \supseteq^- x \cap y$ , as required. Similarly,  $x \cap y \sqsubseteq^+ y$ . Summed up diagrammatically:



(ii) Clearly  $\sqsubseteq$  is reflexive. Supposing  $x \sqsubseteq y$ , i.e.  $x \sqsupseteq^- z \sqsubseteq^+ y$  in  $\mathcal{C}^\infty(A)$  we see that  $x \sqsubseteq^+ y$  and  $y \sqsubseteq^- x$ . Hence if  $x \sqsubseteq y$  and  $y \sqsubseteq x$  in  $\mathcal{C}^\infty(A)$  then  $x$  and  $y$  have the same +ve and -ve events and so are equal. Transitivity follows from (i):



□

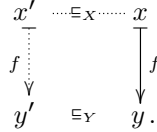
## 7.2 Strategies as presheaves

Let  $A$  be an event structure with polarity. A strategy in  $A$  determines a discrete fibration so a presheaf over the order of finite configurations  $(\mathcal{C}(A), \sqsubseteq_A)$ . In this chapter we only need discrete fibrations over partial orders.

**Definition 52.** A *discrete fibration* over a partial order  $(Y, \sqsubseteq_Y)$  is a partial order  $(X, \sqsubseteq_X)$  and an order-preserving function  $f : X \rightarrow Y$  such that

$$\forall x \in X, y, y' \in Y. y' \sqsubseteq_Y y \ \& \ f(x) = y \implies \exists! x' \sqsubseteq_X x. f(x') = y,$$

as illustrated



**Proposition 53.** Let  $\sigma : S \rightarrow A$  be a pre-strategy in game  $A$ . The map  $\sigma$  taking a finite configuration  $x \in \mathcal{C}(S)$  to  $\sigma x \in \mathcal{C}(A)$  is a discrete fibration from  $(\mathcal{C}(S), \sqsubseteq_S)$  to  $(\mathcal{C}(A), \sqsubseteq_A)$  iff  $\sigma$  is a strategy.

*Proof.* A direct corollary of Proposition 41. □

As discrete fibrations correspond to presheaves, an alternative reading of Proposition 53 is that a pre-strategy  $\sigma : S \rightarrow A$  is a strategy iff  $\sigma$  determines a presheaf over  $(\mathcal{C}(A), \sqsubseteq_A)$ —the presheaf being the functor  $(\mathcal{C}(A), \sqsubseteq_A)^{\text{op}} \rightarrow \mathbf{Set}$  which sends  $y$  to the fibre  $\{x \in \mathcal{C}(S) \mid \sigma x = y\}$  and instances  $y' \sqsubseteq_A y$  to functions from the fibre over  $y$  to the fibre over  $y'$  determined by the fibration.

## 7.3 Strategies as profunctors

A strategy

$$\sigma : A \dashrightarrow B$$

determines a discrete fibration over

$$(\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B}).$$

But

$$(\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B}) \cong (\mathcal{C}(A^\perp), \sqsubseteq_{A^\perp}) \times (\mathcal{C}(B), \sqsubseteq_B) \quad (1)$$

$$\cong (\mathcal{C}(A), \sqsubseteq_A)^{\text{op}} \times (\mathcal{C}(B), \sqsubseteq_B). \quad (2)$$

The first step (1) relies on the correspondence

$$x \leftrightarrow (\{a \mid (1, a) \in x\}, \{b \mid (2, b) \in x\})$$

between a configuration of  $A^\perp \| B$  and a pair, with left component a configuration of  $A^\perp$  and right component a configuration of  $B$ . In the last step (2) we are using the correspondence between configurations of  $A^\perp$  and  $A$  induced by the correspondence  $a \leftrightarrow \bar{a}$  between their events: a configuration  $x$  of  $A^\perp$  corresponds to a configuration  $\bar{x} =_{\text{def}} \{\bar{a} \mid a \in x\}$  of  $A$ . Because  $A^\perp$  reverses the roles of  $+$  and  $-$  in  $A$ , the order  $x \sqsubseteq_{A^\perp} y$  in  $\mathcal{C}(A^\perp)$ ,

$$\begin{array}{ccc} & & y \\ & \swarrow \sqsubseteq & \vdots \text{+} \\ x & \dots \sqsubseteq^- & x \cap y \\ & & \vdots \text{-} \end{array}$$

corresponds to the order  $\bar{y} \sqsubseteq_A \bar{x}$ , i.e.  $\bar{x} \sqsubseteq_A^{\text{op}} \bar{y}$ , in  $\mathcal{C}(A)$ ,

$$\begin{array}{ccc} & & \bar{y} \\ & \swarrow \sqsubseteq & \vdots \text{-} \\ \bar{x} & \dots \sqsubseteq^+ & \bar{x} \cap \bar{y} \\ & & \vdots \text{+} \end{array}$$

It follows that a strategy

$$\sigma : S \rightarrow A^\perp \| B$$

determines a discrete fibration

$$\sigma'' : (\mathcal{C}(S), \sqsubseteq_S) \rightarrow (\mathcal{C}(A), \sqsubseteq_A)^{\text{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$$

where

$$\sigma''(x) = (\overline{\sigma_1 x}, \sigma_2 x),$$

for  $x \in \mathcal{C}(S)$ . The fibration can be viewed as a presheaf over  $(\mathcal{C}(A), \sqsubseteq_A)^{\text{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$ —it assigns the set

$$\{x \in \mathcal{C}(S) \mid \overline{\sigma_1 x} = v \ \& \ \sigma_2 x = z\}$$

to the pair  $(v, z) \in \mathcal{C}(A)^{\text{op}} \times \mathcal{C}(B)$ . One way to define a *profunctor* from  $(\mathcal{C}(A), \sqsubseteq_A)$  to  $(\mathcal{C}(B), \sqsubseteq_B)$  is as a discrete fibration over  $(\mathcal{C}(A), \sqsubseteq_A)^{\text{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$ . Hence the strategy  $\sigma$  determines a profunctor<sup>1</sup>

$$\sigma^{\llcorner} : (\mathcal{C}(A), \sqsubseteq_A) \multimap (\mathcal{C}(B), \sqsubseteq_B).$$

## 7.4 Composition of strategies and profunctors

The operation from strategies  $\sigma$  to profunctors  $\sigma^{\llcorner}$  preserves identities:

**Lemma 54.** *Let  $A$  be an event structure with polarity. For  $x \in \mathcal{C}^\infty(A^\perp \| A)$ ,*

$$x \in \mathcal{C}^\infty(\mathbb{C}_A) \text{ iff } x_2 \sqsubseteq_A \bar{x}_1,$$

where  $x_1 = \{a \in A^\perp \mid (1, a) \in x\}$  and  $x_2 = \{a \in A \mid (2, a) \in x\}$ .

*Proof.* Let  $x \in \mathcal{C}^\infty(A^\perp \| A)$ . From the dependency within copy-cat of the +ve events  $a \in A$  on corresponding -ve events  $\bar{a} \in A^\perp$ , and *vice versa*, as expressed in Proposition 22, we deduce:  $x \in \mathcal{C}^\infty(\mathbb{C}_A)$  iff

$$(i) \ \bar{x}_1^+ \supseteq x_2^+ \quad \text{and} \quad (ii) \ \bar{x}_1^- \subseteq x_2^-,$$

where  $z^+ = \{a \in z \mid \text{pol}_A(a) = +\}$  and  $z^- = \{a \in z \mid \text{pol}_A(a) = -\}$  for  $z \in \mathcal{C}^\infty(A)$ .

It remains to argue that (i) and (ii) iff  $x_2 \supseteq^- \bar{x}_1 \cap x_2 \subseteq^+ \bar{x}_1$ . “*Only if*”: Assume (i) and (ii). Clearly,  $\bar{x}_1 \cap x_2 \subseteq \bar{x}_1$ . Suppose  $a \in \bar{x}_1$  with  $\text{pol}_A(a) = -$ . By (ii),  $a \in x_2$ . Consequently,  $x_1 \cap x_2 \subseteq^+ \bar{x}_1$ . Similarly, (i) entails  $x_2 \supseteq^- \bar{x}_1 \cap x_2$ . “*If*”: To show (i), let  $a \in x_2^+$ . Then as  $x_2 \supseteq^- \bar{x}_1 \cap x_2$  ensures only -ve events are lost in moving from  $x_2$  to  $\bar{x}_1 \cap x_2$ , we see  $a \in \bar{x}_1 \cap x_2$ , so  $a \in \bar{x}_1^+$ . The proof of (ii) is similar.  $\square$

**Corollary 55.** *Let  $A$  be an event structure with polarity. The profunctor  $\gamma_A^{\llcorner}$  of the copy-cat strategy  $\gamma_A$  is an identity profunctor on  $(\mathcal{C}(A), \sqsubseteq_A)$ .*

*Proof.* The profunctor  $\gamma_A^{\llcorner} : (\mathcal{C}(A), \sqsubseteq_A) \multimap (\mathcal{C}(A), \sqsubseteq_A)$  sends  $x \in \mathcal{C}(\mathbb{C}_A)$  to  $(\bar{x}_1, x_2) \in (\mathcal{C}(A), \sqsubseteq_A)^{\text{op}} \times (\mathcal{C}(A), \sqsubseteq_A)$  precisely when  $x_2 \sqsubseteq_A \bar{x}_1$ . It is thus an identity on  $(\mathcal{C}(A), \sqsubseteq_A)$ .  $\square$

We now relate the composition of strategies to the standard composition of profunctors. Let  $\sigma : S \rightarrow A^\perp \| B$  and  $\tau : T \rightarrow B^\perp \| C$  be strategies, so  $\sigma : A \multimap B$  and  $\tau : B \multimap C$ . Abbreviating, for instance,  $(\mathcal{C}(A), \sqsubseteq_A)$  to  $\mathcal{C}(A)$ , strategies  $\sigma$  and  $\tau$  give rise to profunctors  $\sigma^{\llcorner} : \mathcal{C}(A) \multimap \mathcal{C}(B)$  and  $\tau^{\llcorner} : \mathcal{C}(B) \multimap \mathcal{C}(C)$ . Their composition is the profunctor  $\tau^{\llcorner} \circ \sigma^{\llcorner} : \mathcal{C}(A) \multimap \mathcal{C}(C)$  built as a discrete

<sup>1</sup>Most often a profunctor from  $(\mathcal{C}(A), \sqsubseteq_A)$  to  $(\mathcal{C}(B), \sqsubseteq_B)$  is defined as a functor  $(\mathcal{C}(A), \sqsubseteq_A) \times (\mathcal{C}(B), \sqsubseteq_B)^{\text{op}} \rightarrow \mathbf{Set}$ , *i.e.*, as a presheaf over  $(\mathcal{C}(A), \sqsubseteq_A)^{\text{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$ , and as such corresponds to a discrete fibration.

fibration from the discrete fibrations  $\sigma^{\dashv} : \mathcal{C}(S) \rightarrow \mathcal{C}(A)^{\text{op}} \times \mathcal{C}(B)$  and  $\tau^{\dashv} : \mathcal{C}(T) \rightarrow \mathcal{C}(B)^{\text{op}} \times \mathcal{C}(C)$ .

First, we define the set of *matching pairs*,

$$M =_{\text{def}} \{(x, y) \in \mathcal{C}(S) \times \mathcal{C}(T) \mid \sigma_2 x = \overline{\tau_1 y}\},$$

on which we define  $\sim$  as the least equivalence relation for which

$$(x, y) \sim (x', y') \text{ if } \begin{array}{l} x \sqsubseteq_S x' \ \& \ y' \sqsubseteq_T y \ \& \\ \sigma_1 x = \sigma_1 x' \ \& \ \tau_2 y' = \tau_2 y. \end{array}$$

Define an order on equivalence classes  $M / \sim$  by:

$$\begin{aligned} m \sqsubseteq m' \text{ iff } m = \{(x, y)\}_{\sim} \ \& \ m' = \{(x', y')\}_{\sim} \ \& \\ x \sqsubseteq_S x' \ \& \ y \sqsubseteq_T y' \ \& \\ \sigma_2 x = \sigma_2 x' \ \& \ \tau_1 y = \tau_1 y', \end{aligned}$$

for some matching pairs  $(x, y), (x', y')$ —so then  $\sigma_2 x = \sigma_2 x' = \overline{\tau_1 y} = \overline{\tau_1 y'}$ .

*Exercise 56.* Show that  $\sqsubseteq$  above is transitive, so a partial order on  $M / \sim$ . Verify that  $\tau^{\dashv} \circ \sigma^{\dashv}$  is a discrete fibration.  $\square$

**Lemma 57.** *On matching pairs, define*

$$(x, y) \sim_1 (x', y') \text{ iff } \exists s \in S, t \in T. x \overset{s}{\dashv} x' \ \& \ y \overset{t}{\dashv} y' \ \& \ \sigma_2(s) = \overline{\tau_1(t)}.$$

*The smallest equivalence relation including  $\sim_1$  coincides with the relation  $\sim$ .*

*Proof.* From their definitions,  $\sim_1$  is included in  $\sim$ . To prove the converse, it suffices to show that matching pairs  $(x, y), (x', y')$  satisfying

$$\begin{array}{l} x \sqsubseteq_S x' \ \& \ y' \sqsubseteq_T y \ \& \\ \sigma_1 x = \sigma_1 x' \ \& \ \tau_2 y' = \tau_2 y, \end{array}$$

—the clause used in the definition  $\sim$ —are in the equivalence relation generated by  $\sim_1$ . Take a covering chain

$$x \dashv_S x_1 \dashv_S \dots \dashv_S x_m \dashv_S x'$$

in  $(\mathcal{C}(S), \sqsubseteq_S)$ . Here  $\dashv_S$  is the covering relation w.r.t. the order  $\sqsubseteq_S$ , so  $x \dashv_S x_1$  means  $x, x_1$  are distinct and  $x \sqsubseteq_S x_1$  with nothing strictly in between. Via the map  $\sigma$  we obtain

$$\sigma_2 x \dashv_B \sigma_2 x_1 \dashv_B \dots \dashv_B \sigma_2 x_m \dashv_B \sigma_2 x'$$

in  $\mathcal{C}(B)$  where  $\sigma_2 x = \overline{\tau_1 y}$  and  $\sigma_2 x' = \overline{\tau_1 y'}$ . Via the discrete fibration  $\tau^{\dashv}$  we obtain a covering chain in the reverse direction,

$$y \dashv_T y_1 \dashv_T \dots \dashv_T y_m \dashv_T y'$$

in  $(\mathcal{C}(T), \sqsubseteq_T)$ , where each  $(x_i, y_i)$ , for  $1 \leq i \leq m$ , is a matching pair. Moreover,  $(x_i, y_i) \sim_1 (x_{i+1}, y_{i+1})$  at each  $i$  with  $1 \leq i \leq m$ . Hence  $(x, y)$  and  $(x', y')$  are in the equivalence relation generated by  $\sim_1$ .  $\square$

The profunctor composition  $\tau \circ \sigma$  is given as the discrete fibration

$$\tau \circ \sigma : M / \sim \rightarrow \mathcal{C}(A)^{\text{op}} \times \mathcal{C}(C)$$

acting so

$$\{(x, y)\}_{\sim} \mapsto (\overline{\sigma_1 x}, \tau_2 y).$$

It is *not* the case that  $(\tau \circ \sigma)$  and  $\tau \circ \sigma$  coincide up to isomorphism. The profunctor composition  $\tau \circ \sigma$  will generally contain extra equivalence classes  $\{(x, y)\}_{\sim}$  for matching pairs  $(x, y)$  which are “unreachable.” Although  $\sigma_2 x = z = \overline{\tau_1 y}$  automatically for a matching pair  $(x, y)$ , the configurations  $x$  and  $y$  may impose incompatible causal dependencies on their interface  $z$  so never be realized as a configuration in the synchronized composition  $\mathcal{C}(T) \circ \mathcal{C}(S)$ , used in building the composition of strategies  $\tau \circ \sigma$ .

*Example 58.* Let  $A$  and  $C$  both be the empty event structure  $\emptyset$ . Let  $B$  be the event structure consisting of the two concurrent events  $b_1$ , assumed  $-ve$ , and  $b_2$ , assumed  $+ve$  in  $B$ . Let the strategy  $\sigma : \emptyset \rightarrow B$  comprise the event structure  $s_1 \rightarrow s_2$  with  $s_1$   $-ve$  and  $s_2$   $+ve$ ,  $\sigma(s_1) = b_1$  and  $\sigma(s_2) = b_2$ . In  $B^\perp$  the polarities are reversed so there is a strategy  $\tau : B \rightarrow \emptyset$  comprising the event structure  $t_2 \rightarrow t_1$  with  $t_2$   $-ve$  and  $t_1$   $+ve$  yet with  $\tau(t_1) = \overline{b_1}$  and  $\tau(t_2) = \overline{b_2}$ . The equivalence class  $\{(x, y)\}_{\sim}$ , where  $x = \{s_1, s_2\}$  and  $y = \{t_1, t_2\}$ , would be present in the profunctor composition  $\tau \circ \sigma$  whereas  $\tau \circ \sigma$  would be the empty strategy and  $(\tau \circ \sigma)$  accordingly the empty profunctor.  $\square$

**Definition 59.** For  $(x, y)$  a matching pair, define

$$\begin{aligned} x \cdot y =_{\text{def}} & \{(s, *) \mid s \in x \ \& \ \sigma_1(s) \text{ is defined}\} \cup \\ & \{(*, t) \mid t \in y \ \& \ \tau_2(t) \text{ is defined}\} \cup \\ & \{(s, t) \mid s \in x \ \& \ t \in y \ \& \ \sigma_2(s) = \overline{\tau_1(t)}\} \end{aligned}$$

Say  $(x, y)$  is *reachable* if  $x \cdot y \in \mathcal{C}(T) \circ \mathcal{C}(S)$ , and *unreachable* otherwise.

For  $z \in \mathcal{C}(T) \circ \mathcal{C}(S)$  say a *visible prime* of  $z$  is a prime of the form  $[(s, *)]_z$ , for  $(s, *) \in z$ , or  $[(*, t)]_z$ , for  $(*, t) \in z$ .

**Lemma 60.** (i) If  $(x, y)$  is a reachable matching pair and  $(x, y) \sim (x', y')$ , then  $(x', y')$  is a reachable matching pair;  
(ii) For reachable matching pairs  $(x, y)$ ,  $(x', y')$ ,  $(x, y) \sim (x', y')$  iff  $x \cdot y$  and  $x' \cdot y'$  have the same visible primes.

*Proof.* We use the characterization of  $\sim$  in terms of the single-step relation  $\sim_1$  given in Lemma 57.

(i) Suppose  $(x, y) \sim_1 (x', y')$  or  $(x', y') \sim_1 (x, y)$ . By inspection of the construction of the product of stable families in Section 3.3.1, if  $x \cdot y \in \mathcal{C}(T) \circ \mathcal{C}(S)$  then  $x' \cdot y' \in \mathcal{C}(T) \circ \mathcal{C}(S)$ .

(ii) “If”: Suppose  $x \cdot y$  and  $x' \cdot y'$  have the same visible primes, forming the set  $Q$ . Then  $z =_{\text{def}} \bigcup Q \in \mathcal{C}(T) \circ \mathcal{C}(S)$ , being the union of a compatible set of configurations in  $\mathcal{C}(T) \circ \mathcal{C}(S)$ . Moreover,  $z \sqsubseteq x \cdot y, x' \cdot y'$ . Take a covering chain

$$z \xrightarrow{e_1} z_1 \xrightarrow{e_2} z_2 \xrightarrow{e_3} \dots \xrightarrow{e_i} z_i \xrightarrow{e_{i+1}} z_{i+1} \xrightarrow{e_n} x \cdot y$$

in  $\mathcal{C}(T) \circ \mathcal{C}(S)$ . Each  $(\pi_1 z_i, \pi_2 z_i)$  is a matching pair, from the definition of  $\mathcal{C}(T) \circ \mathcal{C}(S)$ . Necessarily,  $e_i = (s_i, t_i)$  for some  $s_i \in S, t_i \in T$ , with  $\sigma_2(s_i) = \overline{\tau_1(t_i)}$ , again by the definition of  $\mathcal{C}(T) \circ \mathcal{C}(S)$ . Thus

$$(\pi_1 z_i, \pi_2 z_i) \sim_1 (\pi_1 z_{i+1}, \pi_2 z_{i+1}).$$

Hence  $z \sim x \cdot y$ , and similarly  $z \sim x' \cdot y'$ , so  $x \cdot y \sim x' \cdot y'$ .

“Only if”: It suffices to observe that if  $(x, y) \sim_1 (x', y')$ , then  $x \cdot y$  and  $x' \cdot y'$  have the same visible primes. But if  $(x, y) \sim_1 (x', y')$  then  $x \cdot y \xrightarrow{(s,t)} x' \cdot y'$ , for some  $s \in S, t \in T$ , and no visible prime in  $x' \cdot y'$  contains  $(s, t)$ .  $\square$

**Lemma 61.** *Let  $\sigma : A \rightarrow B$  and  $\tau : B \rightarrow C$  be strategies. Defining*

$$\varphi_{\sigma, \tau} : \mathcal{C}(T \circ S) \rightarrow M / \sim$$

by

$$\varphi_{\sigma, \tau}(z) = \{(\Pi_1 z, \Pi_2 z)\}_{\sim}$$

yields an injective, order-preserving function from  $(\mathcal{C}(T \circ S), \sqsubseteq_{T \circ S})$  to  $(M / \sim, \sqsubseteq)$ —its range is precisely the equivalence classes  $\{(x, y)\}_{\sim}$  for reachable matching pairs  $(x, y)$ . The diagram

$$\begin{array}{ccc} (\mathcal{C}(T \circ S), \sqsubseteq_{T \circ S}) & \xrightarrow{\varphi_{\sigma, \tau}} & (M / \sim, \sqsubseteq) \\ (\tau \circ \sigma) \llcorner \downarrow & \swarrow \tau \circ \sigma \llcorner & \\ (\mathcal{C}(A), \sqsubseteq_A)^{\text{op}} \times (\mathcal{C}(C), \sqsubseteq_C) & & \end{array}$$

commutes.

*Proof.* For  $z \in \mathcal{C}(T \circ S)$ , we obtain that  $\varphi_{\sigma, \tau}(z) = (\Pi_1 z, \Pi_2 z) = (\pi_1 \cup z, \pi_2 \cup z)$  is a matching pair, from the definition of  $\mathcal{C}(T) \circ \mathcal{C}(S)$ ; it is clearly reachable as  $\pi_1 \cup z \cdot \pi_2 \cup z = \bigcup z \in \mathcal{C}(T) \circ \mathcal{C}(S)$ . For any reachable matching pair  $(x, y)$  let  $z$  be the set of visible primes of  $x \cdot y$ . Then,  $z \in \mathcal{C}(T \circ S)$  and, by Lemma 60(ii),  $(\Pi_1 z, \Pi_2 z) \sim (x, y)$  so  $\varphi_{\sigma, \tau}(z) = \{(x, y)\}_{\sim}$ . Injectivity of  $\varphi_{\sigma, \tau}$  follows directly from Lemma 60(ii).

To show that  $\varphi_{\sigma, \tau}$  is order-preserving it suffices to show if  $z \sqsubseteq z'$  in  $(\mathcal{C}(T \circ S), \sqsubseteq)$  then  $\varphi_{\sigma, \tau}(z) \sqsubseteq \varphi_{\sigma, \tau}(z')$  in  $(M / \sim, \sqsubseteq)$ . (The covering relation  $\sqsubseteq$  is the same as that used in the proof of Lemma 57.) If  $z \sqsubseteq z'$  then either  $z \xrightarrow{p} z'$ , with  $p +ve$ , or  $z' \xrightarrow{p} z$ , with  $p -ve$ , for  $p$  a visible prime of  $\mathcal{C}(T) \circ \mathcal{C}(S)$ , i.e. with  $\max(p)$  of

the form  $(s, *)$  or  $(*, t)$ . We concentrate on the case where  $p$  is +ve (the proof when  $p$  is -ve is similar). In the case where  $p$  is +ve,

$$\Pi_1 z \cdot \Pi_2 z = \bigcup z \sqsubseteq \bigcup z' = \Pi_1 z' \cdot \Pi_2 z'$$

in  $\mathcal{C}(T) \circ \mathcal{C}(S)$  and there is a covering chain

$$\bigcup z = w_0 \xrightarrow{(s_1, t_1) \text{---} c} w_1 \cdots \xrightarrow{(s_n, t_n) \text{---} c} w_n \xrightarrow{\text{max}(p) \text{---} c} \bigcup z'$$

in  $\mathcal{C}(T) \circ \mathcal{C}(S)$ . Each  $w_i$ , for  $0 \leq i \leq m$ , is associated with a reachable matching pair  $(\pi_1 w_i, \pi_2 w_i)$  where  $\pi_1 w_i \cdot \pi_2 w_i = w_i$ . Also  $(\pi_1 w_i, \pi_2 w_i) \sim_1 (\pi_1 w_{i+1}, \pi_2 w_{i+1})$ , for  $0 \leq i < m$ . Hence  $(\Pi_1 z, \Pi_2 z) \sim (\pi_1 w_n, \pi_2 w_n)$ , by Lemma 57(ii). If  $\text{max}(p) = (s, *)$  then  $\pi_1 w_n \xrightarrow{s \text{---} c} \Pi_1 z'$ , with  $s$  +ve, and  $\pi_2 w_n = \Pi_2 z'$ . If  $\text{max}(p) = (*, t)$  then  $\pi_1 w_n = \Pi_1 z'$  and  $\pi_2 w_n \xrightarrow{t \text{---} c} \Pi_2 z'$ , with  $t$  +ve. In either case  $\pi_1 w_n \sqsubseteq_S \Pi_1 z'$  and  $\pi_2 w_n \sqsubseteq_T \Pi_2 z'$  with  $\sigma_2 \pi_1 w_n = \sigma_2 \Pi_1 z'$  and  $\tau_1 \pi_2 w_n = \tau_1 \Pi_2 z'$ . Hence, from the definition of  $\sqsubseteq$  on  $M/\sim$ ,

$$\varphi_{\sigma, \tau}(z) = \{(\Pi_1 z, \Pi_2 z)\}_{\sim} = \{(\pi_1 w_n, \pi_2 w_n)\}_{\sim} \sqsubseteq \{(\Pi_1 z', \Pi_2 z')\}_{\sim} = \varphi_{\sigma, \tau}(z').$$

It remains to show commutativity of the diagram. Let  $z \in \mathcal{C}(T \circ S)$ . Then,

$$(\tau \circ \sigma)(\varphi_{\sigma, \tau}(z)) = (\tau \circ \sigma)(\{(\Pi_1 z, \Pi_2 z)\}_{\sim}) = (\overline{\sigma_1 \Pi_1 z}, \tau_2 \Pi_2 z) = (\tau \circ \sigma)(z),$$

via the definition of  $\tau \circ \sigma$ —as required.  $\square$

Because  $(-)$  does not preserve composition up to isomorphism but only up to the transformation  $\varphi$  of Lemma 61,  $(-)$  forms a *lax* functor from the bicategory of strategies to that of profunctors.

## 7.5 Bistructures

Structure is lost in the passage from an event structure with polarity  $A$  to the order  $(\mathcal{C}(A), \sqsubseteq_A)$ ; it is easy to find two games  $A_1$  comprising purely Player moves and  $A_2$  purely Opponent moves which determine the same order to within isomorphism. However, no essential structure is lost in the move from  $A$  to the ‘bi-order’  $(\mathcal{C}(A), \supseteq^-, \sqsubseteq^+)$ ; the inclusion order  $\sqsubseteq$  is recovered as the transitive closure of  $(\supseteq^-)^{\text{op}} \cup \sqsubseteq^+$  from which the original events can be recovered as primes, and their polarities from the original two orders. In fact, the role of the two orders is to specify polarity.<sup>2</sup> Via Proposition 41, strategies can be expressed order-theoretically as as ‘bi-fibrations,’ discrete fibrations w.r.t. both orders separately. With the introduction of symmetry to event structures we expect category-theoretic analogues of these order-theoretic facts.

<sup>2</sup>Bi-orders  $(\mathcal{C}(A), \supseteq^-, \sqsubseteq^+)$  are clearly related to bistructures [22], with the  $\sqsubseteq^L$  order of bistructures corresponding to  $\supseteq^-$ ,  $\sqsubseteq^R$  to  $\sqsubseteq^+$  and  $\sqsubseteq_A$  to the ‘extensional order’—though there is not yet a full account of the relationship.

# Chapter 8

## Winning ways

What does it mean to win a nondeterministic concurrent game and what is a winning strategy? This chapter extends the work on games and strategies to games with winning conditions and winning strategies.

### 8.1 Winning strategies

A *game with winning conditions* comprises  $G = (A, W)$  where  $A$  is an event structure with polarity and  $W \subseteq \mathcal{C}^\infty(A)$  consists of the *winning configurations* for Player. We define the *losing conditions* to be  $L =_{\text{def}} \mathcal{C}^\infty(A) \setminus W$ . Clearly a game with winning conditions is determined once we specify either its winning or losing conditions, and we can define such a game by specifying its losing conditions.

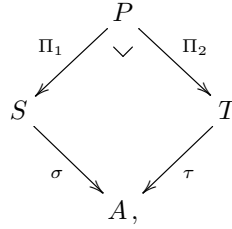
A strategy in  $G$  is a strategy in  $A$ . A strategy in  $G$  is regarded as *winning* if it always prescribes Player moves to end up in a winning configuration, no matter what the activity or inactivity of Opponent. Formally, a strategy  $\sigma : S \rightarrow A$  in  $G$  is *winning (for Player)* if  $\sigma x \in W$  for all +-maximal configurations  $x \in \mathcal{C}^\infty(S)$ —a configuration  $x$  is +-maximal if whenever  $x \xrightarrow{s} c$  then the event  $s$  has -ve polarity. Any achievable position  $z \in \mathcal{C}^\infty(S)$  of the game can be extended to a +-maximal, so winning, configuration (via Zorn's Lemma). So a strategy prescribes Player moves to reach a winning configuration whatever state of play is achieved following the strategy. Note that for a game  $A$ , if winning conditions  $W = \mathcal{C}^\infty(A)$ , *i.e.* every configuration is winning, then any strategy in  $A$  is a winning strategy.

In the special case of a deterministic strategy  $\sigma : S \rightarrow A$  in  $G$  it is winning iff  $\sigma\varphi(x) \in W$  for all  $x \in \mathcal{C}^\infty(S)$ , where  $\varphi$  is the closure operator  $\varphi : \mathcal{C}^\infty(S) \rightarrow \mathcal{C}^\infty(S)$  determined by  $\sigma$  or, equivalently, the images under  $\sigma$  of fixed points of  $\varphi$  lie outside  $L$ . Recall from Section 6.2.3 that a deterministic strategy  $\sigma : S \rightarrow A$  determines a closure operator  $\varphi$  on  $\mathcal{C}^\infty(S)$ : for  $x \in \mathcal{C}^\infty(S)$ ,

$$\varphi(x) = x \cup \{s \in S \mid \text{pol}(s) = + \ \& \ \text{Neg}[\{s\}] \subseteq x\}.$$

Clearly, we can equivalently say a strategy  $\sigma : S \rightarrow A$  in  $G$  is winning if it always prescribes Player moves to avoid ending up in a losing configuration, no matter what the activity or inactivity of Opponent; a strategy  $\sigma : S \rightarrow A$  in  $G$  is winning if  $\sigma x \notin L$  for all +-maximal configurations  $x \in \mathcal{C}^\infty(S)$

Informally, we can also understand a strategy as winning for Player if when played against any counter-strategy of Opponent, the final result is a win for Player. Suppose  $\sigma : S \rightarrow A$  is a strategy in a game  $(A, W)$ . A counter-strategy is strategy of Opponent, so a strategy  $\tau : T \rightarrow A^\perp$  in the dual game. We can view  $\sigma$  as a strategy  $\sigma : \emptyset \rightarrow A$  and  $\tau$  as a strategy  $\tau : A \rightarrow \emptyset$ . Their composition  $\tau \circ \sigma : \emptyset \rightarrow \emptyset$  is not in itself so informative. Rather it is the status of the configurations in  $\mathcal{C}^\infty(A)$  their full interaction induces which decides which of Player or Opponent wins. Ignoring polarities, we have total maps of event structures  $\sigma : S \rightarrow A$  and  $\tau : T \rightarrow A$ . Form their pullback,



to obtain the event structure  $P$  resulting from the interaction of  $\sigma$  and  $\tau$ . (Note  $P \cong \text{Pr}(\mathcal{C}(T) \circ \mathcal{C}(S))$ , in the terms of Chapter 4, by the remarks of Section 4.3.3.) Because  $\sigma$  or  $\tau$  may be nondeterministic there can be more than one maximal configuration  $z$  in  $\mathcal{C}^\infty(P)$ . A maximal configuration  $z$  in  $\mathcal{C}^\infty(P)$  images to a configuration  $\sigma \Pi_1 z = \tau \Pi_2 z$  in  $\mathcal{C}^\infty(A)$ . Define the set of *results* of the interaction of  $\sigma$  and  $\tau$  to be

$$\langle \sigma, \tau \rangle =_{\text{def}} \{ \sigma \Pi_1 z \mid z \text{ is maximal in } \mathcal{C}^\infty(P) \}.$$

We shall show the strategy  $\sigma$  is a winning for Player iff all the results of the interaction  $\langle \sigma, \tau \rangle$  lie within the winning configurations  $W$ , for any counter-strategy  $\tau : T \rightarrow A^\perp$  of Opponent.

It will be convenient later to have proved facts about +-maximality in the broader context of the composition of arbitrary strategies.

*Convention 62.* Refer to the construction of the composition of pre-strategies  $\sigma : S \rightarrow A^\perp \parallel B$  and  $\tau : B^\perp \parallel C$  in Chapter 4 We shall say a configuration  $x$  of either  $\mathcal{C}^\infty(S)$ ,  $\mathcal{C}^\infty(T)$  or  $(\mathcal{C}(T) \circ \mathcal{C}(S))^\infty$  is +-maximal if whenever  $x \xrightarrow{e} c$  then the event  $e$  has -ve polarity. (In the case of  $(\mathcal{C}(T) \circ \mathcal{C}(S))^\infty$  an event of -ve polarity is deemed to be one of the form  $(s, *)$ , with  $s$  -ve in  $S$ , or  $(*, t)$ , with  $t$  -ve in  $T$ .)

**Lemma 63.** *Let  $\sigma : S \rightarrow A^\perp \parallel B$  and  $\tau : T \rightarrow B^\perp \parallel C$  be receptive pre-strategies. Then,*

$$\begin{aligned}
 & z \in (\mathcal{C}(T) \circ \mathcal{C}(S))^\infty \text{ is +-maximal iff} \\
 & \pi_1 z \in \mathcal{C}^\infty(S) \text{ is +-maximal \& } \pi_2 z \in \mathcal{C}^\infty(T) \text{ is +-maximal.}
 \end{aligned}$$

*Proof.* Let  $z \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ . “*Only if*”: Assume  $z$  is +-maximal. Suppose, for instance,  $\pi_1 z$  is not +-maximal. Then,  $\pi_1 \xrightarrow{s} \_$  for some +ve event  $s \in S$ . Consider the two cases. *Case*  $\sigma_1(s)$  *is defined*: Form the configuration  $z \cup \{(s, *)\} \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ , to contradict the +-maximality of  $z$ . *Case*  $\sigma_2(s)$  *is defined*: As  $s$  is +ve by the receptivity of  $\tau$  there is  $t \in T$  such that  $\tau_1(t) = \sigma_2(s)$ . Form the configuration  $z \cup \{(s, t)\} \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ , to contradict the +-maximality of  $z$ . The argument showing  $\pi_2 z$  is +-maximal is similar.

“*If*”: Assume both  $\pi_1 z$  and  $\pi_2 z$  are +-maximal. Suppose  $z$  were not +-maximal. Then, either

- $z \xrightarrow{(s, *)} \_$  or  $z \xrightarrow{(s, t)} \_$  with  $s$  is a +ve event of  $S$ , or
- $z \xrightarrow{(*, t)} \_$  or  $z \xrightarrow{(s, t)} \_$  with  $t$  a +ve event of  $T$ .

But then either  $\pi_1 z \xrightarrow{s} \_$ , contradicting the +-maximality of  $\pi_1 z$ , or  $\pi_2 z \xrightarrow{t} \_$ , contradicting the +-maximality of  $\pi_2 z$ .  $\square$

**Lemma 64.** *Let  $\sigma : S \rightarrow A$  be a strategy in a game  $(A, W)$ . The strategy  $\sigma$  is a winning for Player iff  $\langle \sigma, \tau \rangle \subseteq W$  for all (deterministic) strategies  $\tau : T \rightarrow A^\perp$ .*

*Proof.* “*Only if*”: Suppose  $\sigma$  is winning, i.e.  $\sigma x \in W$  for all +-maximal  $x \in \mathcal{C}^\infty(S)$ . Let  $\tau : T \rightarrow A^\perp$  be a strategy. By Lemma 63,

$$\begin{aligned} & z \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty \text{ is +-maximal} \\ \text{iff} \\ & \pi_1 z \in \mathcal{C}^\infty(S) \text{ is +-maximal \& } \pi_2 z \in \mathcal{C}^\infty(T) \text{ is +-maximal.} \end{aligned}$$

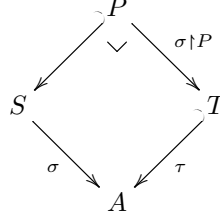
Note  $x \in \mathcal{C}^\infty(\text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)))$  is +-maximal iff  $\cup x \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$  is +-maximal. Letting  $x$  be maximal in  $\mathcal{C}^\infty(\text{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)))$  it is certainly +-maximal. Hence  $\cup x$  is +-maximal in  $(\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ , whence  $\Pi_1 x = \pi_1 \cup x$  is +-maximal in  $\mathcal{C}^\infty(S)$ . It follows that  $\sigma \Pi_1 x \in W$  as  $\sigma$  is winning. Hence  $\langle \sigma, \tau \rangle \subseteq W$ .

“*If*”: Assume  $\langle \sigma, \tau \rangle \subseteq W$  for all strategies  $\tau : T \rightarrow A^\perp$ . Suppose  $x$  is +-maximal in  $\mathcal{C}^\infty(S)$ . Define  $T$  to be the event structure given as the restriction

$$T =_{\text{def}} A^\perp \upharpoonright \sigma x \cup \{a \in A^\perp \mid \text{pol}_{A^\perp} = -\}.$$

Let  $\tau : T \rightarrow A^\perp$  be the inclusion map  $T \hookrightarrow A^\perp$ . The pre-strategy  $\tau$  can be checked to be receptive and innocent, so a strategy. (In fact,  $\tau$  is a *deterministic* strategy as all its +ve events lie within the configuration  $\sigma x$ .) One way to describe a

pullback of  $\tau$  along  $\sigma$  is as the “inverse image”  $P =_{\text{def}} S \upharpoonright \{s \in S \mid \sigma(s) \in T\}$ :



From the definition of  $T$  and  $P$  we see  $x \in \mathcal{C}^\infty(P)$ ; and moreover that  $x$  is maximal in  $\mathcal{C}^\infty(P)$  as  $x$  is  $+$ -maximal in  $\mathcal{C}^\infty(S)$ . Hence  $\sigma x \in \langle \sigma, \tau \rangle$  ensuring  $\sigma x \in W$ , as required.

The proof is unaffected if we restrict to *deterministic* counter-strategies  $\tau : T \rightarrow A^\perp$ .  $\square$

**Corollary 65.** *There are the following four equivalent ways to say that a strategy  $\sigma : S \rightarrow A$  is winning in  $(A, W)$ —we write  $L$  for the losing configurations  $\mathcal{C}^\infty(A) \setminus W$ :*

1.  $\sigma x \in W$  for all  $+$ -maximal configurations  $x \in \mathcal{C}^\infty(S)$ , i.e. the strategy prescribes Player moves to reach a winning configuration, no matter what the activity or inactivity of Opponent;
2.  $\sigma x \notin L$  for all  $+$ -maximal configurations  $x \in \mathcal{C}^\infty(S)$ , i.e. the strategy prescribes Player moves to avoid ending up in a losing configuration, no matter what the activity or inactivity of Opponent;
3.  $\langle \sigma, \tau \rangle \subseteq W$  for all strategies  $\tau : T \rightarrow A^\perp$ , i.e. all plays against counter-strategies of the Opponent result in a win for Player;
4.  $\langle \sigma, \tau \rangle \subseteq W$  for all deterministic strategies  $\tau : T \rightarrow A^\perp$ , i.e. all plays against deterministic counter-strategies of the Opponent result in a win for Player.

Not all games with winning conditions have winning strategies. Consider the game  $A$  consisting of one player move  $\oplus$  and one opponent move  $\ominus$  inconsistent with each other, with  $\{\{\oplus\}\}$  as its winning conditions. This game has no winning strategy; any strategy  $\sigma : S \rightarrow A$ , being receptive, will have an event  $s \in S$  with  $\sigma(s) = \ominus$ , and so the losing  $\{s\}$  as a  $+$ -maximal configuration.

## 8.2 Operations

### 8.2.1 Dual

There is an obvious dual of a game with winning conditions  $G = (A, W_G)$ :

$$G^\perp = (A^\perp, W_{G^\perp})$$

where, for  $x \in \mathcal{C}^\infty(A)$ ,

$$x \in W_{G^\perp} \text{ iff } \bar{x} \notin W_G.$$

We are using the notation  $a \leftrightarrow \bar{a}$ , giving the correspondence between events of  $A$  and  $A^\perp$ , extended to their configurations:  $\bar{x} =_{\text{def}} \{\bar{a} \mid a \in x\}$ , for  $x \in \mathcal{C}^\infty(A)$ . As usual the dual reverses the roles of Player and Opponent and correspondingly the roles of winning and losing conditions.

### 8.2.2 Parallel composition

The parallel composition of two games with winning conditions  $G = (A, W_G)$ ,  $H = (B, W_H)$  is

$$G \parallel H =_{\text{def}} (A \parallel B, W_G \parallel \mathcal{C}^\infty(B) \cup \mathcal{C}^\infty(A) \parallel W_H)$$

where  $X \parallel Y = \{\{1\} \times x \cup \{2\} \times y \mid x \in X \ \& \ y \in Y\}$  when  $X$  and  $Y$  are subsets of configurations. In other words, for  $x \in \mathcal{C}^\infty(A \parallel B)$ ,

$$x \in W_{G \parallel H} \text{ iff } x_1 \in W_G \text{ or } x_2 \in W_H,$$

where  $x_1 = \{a \mid (1, a) \in x\}$  and  $x_2 = \{b \mid (2, b) \in x\}$ . To win in  $G \parallel H$  is to win in either game. Its losing conditions are  $L_A \parallel L_B$ —to lose is to lose in both games  $G$  and  $H$ .<sup>1</sup> The unit of  $\parallel$  is  $(\emptyset, \emptyset)$ . In order to disambiguate the various forms of parallel composition, we shall sometimes use the linear-logic notation  $G \wp H$  for the parallel composition  $G \parallel H$  of games with winning strategies.

### 8.2.3 Tensor

Defining  $G \otimes H =_{\text{def}} (G^\perp \parallel H^\perp)^\perp$  we obtain a game where to win is to win in both games  $G$  and  $H$ —so to lose is to lose in either game. More explicitly,

$$(A, W_A) \otimes (B, W_B) =_{\text{def}} (A \parallel B, W_A \parallel W_B).$$

The unit of  $\otimes$  is  $(\emptyset, \{\emptyset\})$ .

### 8.2.4 Function space

With  $G \multimap H =_{\text{def}} G^\perp \parallel H$  a win in  $G \multimap H$  is a win in  $H$  conditional on a win in  $G$ .

**Proposition 66.** *Let  $G = (A, W_G)$  and  $H = (B, W_H)$  be games with winning conditions. Write  $W_{G \multimap H}$  for the winning conditions of  $G \multimap H$ , so  $G \multimap H = (A^\perp \parallel B, W_{G \multimap H})$ . For  $x \in \mathcal{C}^\infty(A^\perp \parallel B)$ ,*

$$x \in W_{G \multimap H} \text{ iff } \bar{x}_1 \in W_G \implies x_2 \in W_H.$$

<sup>1</sup>I'm grateful to Nathan Bowler, Pierre Clairambault and Julian Gutierrez for guidance in the definition of parallel composition of games with winning conditions.

*Proof.* Letting  $x \in \mathcal{C}^\infty(A^\perp \| B)$ ,

$$\begin{aligned} x \in W_{G \rightarrow H} &\text{ iff } x \in W_{G^\perp \| H} \\ &\text{ iff } x_1 \in W_{G^\perp} \text{ or } x_2 \in W_H \\ &\text{ iff } \overline{x_1} \notin W_G \text{ or } x_2 \in W_H \\ &\text{ iff } \overline{x_1} \in W_G \implies x_2 \in W_H. \end{aligned}$$

□

### 8.3 The bicategory of winning strategies

We can again follow Joyal and define strategies between games now with winning conditions: a (winning) strategy from  $G$ , a game with winning conditions, to another  $H$  is a (winning) strategy in  $G \multimap H = G^\perp \| H$ . We compose strategies as before. We first show that the composition of winning strategies is winning.

**Lemma 67.** *Let  $\sigma$  be a winning strategy in  $G^\perp \| H$  and  $\tau$  be a winning strategy in  $H^\perp \| K$ . Their composition  $\tau \circ \sigma$  is a winning strategy in  $G^\perp \| K$ .*

*Proof.* Let  $G = (A, W_G)$ ,  $H = (B, W_H)$  and  $K = (C, W_K)$ .

Suppose  $x \in \mathcal{C}^\infty(T \circ S)$  is +-maximal. Then  $\bigcup x \in (\mathcal{C}(T) \circ \mathcal{C}(S))^\infty$ . By Zorn's Lemma we can extend  $\bigcup x$  to a maximal configuration  $z \supseteq \bigcup x$  in  $(\mathcal{C}(T) \circ \mathcal{C}(S))^\infty$  with the property that all events of  $z \setminus \bigcup x$  are synchronizations of the form  $(s, t)$  for  $s \in S$  and  $t \in T$ . Then,  $z$  will be +-maximal in  $(\mathcal{C}(T) \circ \mathcal{C}(S))^\infty$  with

$$\sigma_1 \pi_1 z = \sigma_1 \pi_1 \bigcup x \quad \& \quad \tau_2 \pi_2 z = \tau_2 \pi_2 \bigcup x. \quad (1)$$

By Lemma 63,

$$\pi_1 z \text{ is +-maximal in } S \quad \& \quad \pi_2 z \text{ is +-maximal in } T.$$

As  $\sigma$  and  $\tau$  are winning,

$$\sigma \pi_1 z \in W_{G^\perp \| H} \quad \& \quad \tau \pi_2 z \in W_{H^\perp \| K}.$$

Now  $\sigma \pi_1 z \in W_{G^\perp \| H}$  expresses that

$$\overline{\sigma_1 \pi_1 z} \in W_G \implies \sigma_2 \pi_1 z \in W_H \quad (2)$$

and  $\tau \pi_2 z \in W_{H^\perp \| K}$  that

$$\overline{\tau_1 \pi_2 z} \in W_H \implies \tau_2 \pi_2 z \in W_K, \quad (3)$$

by Proposition 66. But  $\sigma_2 \pi_1 z = \overline{\tau_1 \pi_2 z}$ , so (2) and (3) yield

$$\overline{\sigma_1 \pi_1 z} \in W_G \implies \tau_2 \pi_2 z \in W_K.$$

By (1)

$$\overline{\sigma_1 \pi_1 \bigcup x} \in W_G \implies \tau_2 \pi_2 \bigcup x \in W_K,$$

*i.e.* by Proposition 23,

$$\overline{v_1 x} \in W_G \implies v_2 x \in W_K$$

in the span of the composition  $\tau \circ \sigma$ . Hence  $x \in W_{G^\perp \| K}$ , as required. □

For a general game with winning conditions  $(A, W)$  the copy-cat strategy need not be winning, as shown in the following example.

*Example 68.* Let  $A$  consist of two events, one +ve event  $\oplus$  and one -ve event  $\ominus$ , inconsistent with each other. Take as winning conditions the set  $\{\{\oplus\}\}$ . The event structure  $\mathbb{C}_A$ :

$$\begin{array}{c} A^\perp \ominus \rightarrow \oplus A \\ \oplus \leftarrow \ominus \end{array}$$

To see  $\mathbb{C}_A$  is not winning consider the configuration  $x$  consisting of the two -ve events in  $\mathbb{C}_A$ . Then  $x$  is +-maximal as any +ve event is inconsistent with  $x$ . However,  $\bar{x}_1 \in W$  while  $x_2 \notin W$ , failing the winning condition of  $(A, W) \dashv (A, W)$ .

Recall from Chapter 7, that each event structure with polarity  $A$  possesses a Scott order on its configurations  $\mathcal{C}^\infty(A)$ :

$$x' \sqsubseteq x \text{ iff } x' \supseteq^- x \cap x' \sqsubseteq^+ x.$$

A necessary and sufficient for copy-cat to be winning w.r.t. a game  $(A, W)$ :

$$\forall x, x' \in \mathcal{C}^\infty(A). \text{ if } x' \sqsubseteq x \text{ \& } x' \text{ is +-maximal \& } x \text{ is --maximal,} \quad \text{(Cwins)} \\ \text{then } x \in W \implies x' \in W.$$

**Lemma 69.** *Let  $(A, W)$  be a game with winning conditions. The copy-cat strategy  $\gamma_A : \mathbb{C}_A \rightarrow A^\perp \parallel A$  is winning iff  $(A, W)$  satisfies (Cwins).*

*Proof.* By Lemma 54,

$$z \in \mathcal{C}^\infty(\mathbb{C}_A) \text{ iff } z = \{1\} \times \bar{x} \cup \{2\} \times x' \text{ with } x' \sqsubseteq_A x,$$

for  $x, x' \in \mathcal{C}^\infty(A)$ . In this situation  $z$  is +-maximal iff both  $x$  is --maximal and  $x'$  is +maximal. Thus (Cwins) expresses precisely that copy-cat is winning.  $\square$

A robust sufficient condition on an event structure with polarity  $A$  which ensures that copy-cat is a winning strategy for all choices of winning conditions is the property

$$\forall x \in \mathcal{C}(A). x \xrightarrow{a} \text{c} \text{ \& } x \xrightarrow{a'} \text{c} \text{ \& } \text{pol}(a) = + \text{ \& } \text{pol}(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A). \quad (\ddagger)$$

This property, which says immediate conflict respects polarity, is seen earlier in Lemma 45 (characterizing those  $A$  for which copy-cat is deterministic).

**Proposition 70.** *Let  $A$  be an event structure with polarity. Copy-cat is a winning strategy for all games  $(A, W)$  with winning conditions  $W$  iff  $A$  satisfies  $(\ddagger)$ .*

*Proof.* “If”: Assume  $(\ddagger)$ . Let  $W \subseteq \mathcal{C}^\infty(A)$ . We show (Cwins) holds for the game with winning conditions  $(A, W)$ . For  $x, x' \in \mathcal{C}^\infty(A)$ , assume

$$x' \sqsubseteq x \text{ \& } x' \text{ is +-maximal \& } x \text{ is --maximal.}$$

Then, as  $x' \supseteq^- x \cap x' \sqsubseteq^+ x$ , there are covering chains associated with purely +ve and -ve events from  $x \cap x'$  to  $x$  and  $x'$ , respectively:

$$\begin{aligned} x \cap x' &\xrightarrow{+} \dots \xrightarrow{+} x, \\ x \cap x' &\xrightarrow{-} \dots \xrightarrow{-} x'. \end{aligned}$$

If one of the covering chains is of zero length then so must the other be—otherwise we contradict one or other of the maximality assumptions. On the other hand, if both are nonempty, by repeated use of  $(\ddagger)$  we again contradict a maximality assumption, *e.g.*

$$\begin{array}{ccccccc} y_1 & \xrightarrow{+} & x_1 \cup x'_1 & \xrightarrow{+} & \dots & \xrightarrow{+} & x \cup x'_1 \\ -\Uparrow & & -\Uparrow & & & & -\Uparrow \\ x \cap x' & \xrightarrow{+} & x_1 & \xrightarrow{+} & \dots & \xrightarrow{+} & x \end{array}$$

shows how a repeated use of  $(\ddagger)$  contradicts the  $--$ -maximality of  $x$ . We conclude  $x = x \cap x' = x'$  so certainly  $x \in W \implies x' \in W$ , as required to fulfil **(Cwins)**.

“*Only if*”: Suppose  $A$  failed  $(\ddagger)$ , *i.e.*  $x \xrightarrow{a} x_1$  &  $x \xrightarrow{a'} x_2$  with  $x_1 \not\uparrow x_2$  and  $pol_A(a) = +$  and  $pol(a') = -$  within the finite configurations of  $A$ . The set  $\{1\} \times \bar{x}_1 \cup \{2\} \times x_2$  is certainly a finite configuration of  $A^\perp \parallel A$  and is easily checked to also be a configuration of  $\mathbb{C}_A$ . Define winning conditions by

$$W = \{x \in \mathcal{C}^\infty(A) \mid a \in x\}.$$

Let  $z \in \mathcal{C}^\infty(\mathbb{C}_A)$  be a  $+$ -maximal extension of  $\{1\} \times \bar{x}_1 \cup \{2\} \times x_2$  (the maximal extension exists by Zorn’s Lemma). Take  $z_1 = \{a \mid (1, a) \in z\}$  and  $z_2 = \{a \mid (2, a) \in z\}$ . Then  $\bar{z}_1 \supseteq x_1$  and  $z_2 \supseteq x_2$ . As  $a \in \bar{z}_1$  we obtain  $\bar{z}_1 \in W$ , whereas  $z_2 \notin W$  because  $z_2$  extends  $y$  which is inconsistent with  $a$ . Hence copy-cat is not winning in  $(A, W)^\perp \parallel (A, W)$ .  $\square$

We can now refine the bicategory of strategies **Games** to the bicategory **WGames** with objects games with winning conditions  $G, H, \dots$  satisfying **(Cwins)** and arrows winning strategies  $G \twoheadrightarrow H$ ; 2-cells, their vertical and horizontal composition is as before. Its restriction to deterministic strategies yields a bicategory **WDGames** equivalent to a simpler order-enriched category.

## 8.4 Total strategies

As an application of winning conditions we apply them to pick out a subcategory of “total strategies,” informally strategies in which Player can always answer a move of Opponent.<sup>2</sup>

We restrict attention to ‘simple games’ (games and strategies are alternating and begin with opponent moves—see Section 6.2.4). Here a strategy is *total* if

<sup>2</sup>This section is inspired by [23], though differs in several respects.

all its finite maximal sequences are even, so ending in a +ve move, *i.e.* a move of Player. In general, the composition of total strategies need not be total—see the Exercise below. However, as we will see, we can pick out a subcategory of ‘simple games’ with suitable winning conditions. Within this full subcategory of games with winning conditions winning strategies will be total and moreover compose.

*Exercise 71.* Exhibit two total strategies whose composition is not total.  $\square$

As objects of the subcategory we choose simple games with winning strategies,

$$(A, W_A)$$

where  $A$  is a simple game and  $W_A$  is a subset of possibly infinite sequences  $s_1 s_2 \dots$  satisfying

$$W_A \cap \text{Finite}(A) = \text{Even}(A) \quad (\mathbf{Tot})$$

*i.e.* the finite sequences in  $W_A$  are precisely those of even length. Note that winning strategies in such a game will be total. (Below we use ‘sequence’ to mean allowable finite or infinite sequences of the appropriate simple game.)

The function space  $(A, W_A) \multimap (B, W_B)$ , given as  $(A, W_A)^\perp \parallel (B, W_B)$ , has winning conditions  $W$  such that

$$s \in W \text{ iff } s \upharpoonright A \in W_A \implies s \upharpoonright B \in W_B.$$

**Lemma 72.** *For  $s$  a sequence of  $A^\perp \parallel B$ ,  $s$  is even iff  $s \upharpoonright A$  is odd or  $s \upharpoonright B$  is even.*

*Proof.* By parity, considering the final move of the sequence.

“*Only if*”: Assume  $s$  is even, *i.e.* its final event is +ve. If  $s$  ends in  $B$ ,  $s \upharpoonright B$  ends in + so is even. If  $s$  ends in  $A$ ,  $s \upharpoonright A$  ends in – so is odd.

“*If*”: Assume  $s \upharpoonright A$  is odd or  $s \upharpoonright B$  is even. Suppose, to obtain a contradiction, that  $s$  is not even, *i.e.*  $s$  is odd so ends in –. If  $s$  ends in  $B$ ,  $s \upharpoonright B$  ends in – so is odd and consequently  $s \upharpoonright A$  even (as the length of  $s$  is the sum of the lengths of  $s \upharpoonright A$  and  $s \upharpoonright B$ ). Similarly, if  $s$  ends in  $A$ ,  $s \upharpoonright A$  ends in + so  $s \upharpoonright A$  is even and  $s \upharpoonright B$  is odd. Either case contradicts the initial assumption. Hence  $s$  is even.  $\square$

It follows that  $W$ , the winning conditions of the function space, satisfies **(Tot)**: Let  $s$  be a finite sequence of a strategy in  $A^\perp \parallel B$ . Then,

$$\begin{aligned} s \in W \text{ iff } s \upharpoonright A \in W_A &\implies s \upharpoonright B \in W_B \\ &\text{iff } s \upharpoonright A \notin W_A \text{ or } s \upharpoonright B \in W_B \\ &\text{iff } s \upharpoonright A \text{ is odd or } s \upharpoonright B \text{ is even} \\ &\text{iff } s \text{ is even.} \end{aligned}$$

All maps in the subcategory (which are winning strategies in its function spaces  $(A, W_A) \multimap (B, W_B)$ ) compose (because winning strategies do) and are total (because winning conditions of its function spaces satisfy **(Tot)**).

## 8.5 On determined games

A game with winning conditions  $G$  is said to be *determined* when either Player or Opponent has a winning strategy, *i.e.* either there is a winning strategy in  $G$  or in  $G^\perp$ .<sup>3</sup> Not all games are determined. Neither the game  $G$  consisting of one player move  $\oplus$  and one opponent move  $\ominus$  inconsistent with each other, with  $\{\{\oplus\}\}$  as winning conditions, nor the game  $G^\perp$  have a winning strategy.

*Notation 73.* Let  $\sigma : S \rightarrow A$  be a strategy. We say  $y \in \mathcal{C}^\infty(A)$  is  $\sigma$ -*reachable* iff  $y = \sigma x$  for some  $x \in \mathcal{C}^\infty(S)$ . Let  $y' \subseteq y$  in  $\mathcal{C}^\infty(A)$ . Say  $y'$  is *--maximal in  $y$*  iff  $y \text{---}^- y''$  implies  $y'' \not\subseteq y$ . Similarly, say  $y'$  is *+maximal in  $y$*  iff  $y \text{---}^+ y''$  implies  $y'' \not\subseteq y$ .

**Lemma 74.** *Let  $(A, W)$  be a game with winning conditions. Let  $y \in \mathcal{C}(A)$ . Suppose*

$$\begin{aligned} & \forall y' \in \mathcal{C}(A). \\ & y' \subseteq y \ \& \ y' \text{ is --maximal in } y \ \& \ \text{not +maximal in } y \\ & \implies \\ & \{y'' \in \mathcal{C}(A) \mid y' \subseteq^+ y'' \ \& \ (y'' \setminus y') \cap y = \emptyset\} \cap W = \emptyset. \end{aligned}$$

*Then  $y$  is  $\sigma$ -reachable in all winning strategies  $\sigma$ .*

*Proof.* Assume the property above of  $y \in \mathcal{C}(A)$ . Suppose, to obtain a contradiction, that  $y$  is not  $\sigma$ -reachable in a winning strategy  $\sigma : S \rightarrow A$ .

As  $y$  is finite, there is a maximal  $y' \in \mathcal{C}(A)$  with  $y' \not\subseteq y$  and  $y'$   $\sigma$ -reachable.

By receptivity of  $\sigma$ , the configuration  $y'$  is --maximal in  $y$ . As  $y' \not\subseteq y$ , we must therefore have  $y' \text{---}^+ y_0 \subseteq y$  in  $\mathcal{C}(A)$ , *i.e.*  $y'$  is not +maximal in  $y$ . From the property assumed of  $y$  we deduce

$$y' \notin W \ \& \ \forall y'' \in W. y' \subseteq^+ y'' \implies (y'' \setminus y') \cap y \neq \emptyset.$$

As  $y'$  is  $\sigma$ -reachable,  $y' = \sigma x'$  for some  $x' \in \mathcal{C}(S)$ . As  $\sigma$  is winning, there is +-maximal extension  $x' \subseteq^+ x''$  in  $\mathcal{C}(S)$  such that  $\sigma x'' \in W$ . Hence

$$(\sigma x'' \setminus y') \cap y \neq \emptyset.$$

Taking a  $\leq_A$ -minimal event  $a_1$  in the above set we obtain

$$y' \text{---}^{a_1} y_1 \subseteq^+ \sigma x''.$$

By Proposition 41,  $y_1 = \sigma x_1$  for some  $x_1 \in \mathcal{C}(S)$  with  $x' \text{---}^+ x_1 \subseteq x''$ . But this contradicts the choice of  $y'$  as the maximal reachable sub-configuration of  $y$ . Hence the original assumption that  $y$  is not  $\sigma$ -reachable must be false.  $\square$

Recall the property  $(\ddagger)$  of an event structure with polarity  $A$ , first seen in Lemma 45, though here rephrased a little:

$$\forall y, y_1, y_2 \in \mathcal{C}(A). y \text{---}^- y_1 \ \& \ y \text{---}^+ y_2 \implies y_1 \uparrow y_2. \quad (\ddagger)$$

<sup>3</sup>This section is based on work with Julian Gutierrez.

**Corollary 75.** *If  $A$ , an event structure with polarity, fails to satisfy  $(\ddagger)$ , then there are winning conditions  $W$ , for which the game  $(A, W)$  is not determined.*

*Proof.* Suppose  $(\ddagger)$  failed, that  $y \overset{-}{\dashv} y_1$  and  $y \overset{+}{\dashv} y_2$  and  $y_1 \uparrow y_2$  in  $\mathcal{C}(A)$ . Assign configurations  $\mathcal{C}^\infty(A)$  to winning conditions  $W$  or its complement as follows:

- (i) for  $y''$  with  $y_1 \subseteq^+ y''$ , assign  $y'' \notin W$ ;
- (ii) for  $y''$  with  $y_2 \subseteq^- y''$ , assign  $y'' \in W$ ;
- (iii) for  $y''$  with  $y' \subseteq^+ y''$  and  $(y'' \setminus y') \cap y = \emptyset$ , for some sub-configuration  $y'$  of  $y$  with  $y'$  --maximal and not +-maximal in  $y$ , assign  $y'' \notin W$ ;
- (iv) for  $y''$  with  $y' \subseteq^- y''$  and  $(y'' \setminus y') \cap y = \emptyset$ , for some sub-configuration  $y'$  of  $y$  with  $y'$  +-maximal and not --maximal in  $y$ , assign  $y'' \in W$ ;
- (v) assign arbitrarily in all other cases.

We should check the assignment is well-defined, that we do not assign a configuration both to  $W$  and its complement.

Clearly the first two cases (i) and (ii) are disjoint as  $y_1 \uparrow y_2$ .

The two cases (iii) and (iv) are also disjoint. Suppose otherwise, that both (iii) and (iv) hold for  $y''$ , viz.

$$\begin{aligned} y'_1 \subseteq^+ y'' \ \& \ (y'' \setminus y'_1) \cap y = \emptyset \ \& \\ & y'_1 \text{ is --maximal \ \& \ not +-maximal in } y, \ \text{and} \\ y'_2 \subseteq^- y'' \ \& \ (y'' \setminus y'_2) \cap y = \emptyset \ \& \\ & y'_2 \text{ is +-maximal \ \& \ not --maximal in } y. \end{aligned}$$

As

$$y'_1 \subseteq^+ y'' \supseteq^- y'_2$$

we deduce  $y'_2 \subseteq^- y'_1$ , i.e. all the -- events of  $y'_2$  are in  $y'_1$ . Now let  $a \in y'_2$ . Then  $a \in y$  as  $y'_2 \subseteq y$ . Therefore  $a \notin y'' \setminus y'_1$ , by assumption. But  $a \in y''$  as  $y'_2 \subseteq^- y''$ , so  $a \in y'_1$ . We conclude  $y'_2 \subseteq y'_1$ . A similar dual argument shows  $y'_1 \subseteq y'_2$ . Thus  $y'_1 = y'_2$ . But this implies that  $y'_1$  is both --maximal and not --maximal in  $y$  — a contradiction.

Suppose both the conditions (i) and (iv) are met by  $y''$ . From (vi), as  $y'$  is +-maximal & not --maximal in  $y$ ,

$$y' \overset{a}{\dashv} y_0 \subseteq y,$$

for some event  $a$  with  $\text{pol}_A(a) = -$  and  $y_0 \in \mathcal{C}^\infty(A)$ . From (i),  $y \subseteq y''$ , so

$$y' \overset{a}{\dashv} y_0 \subseteq y''.$$

Therefore

$$a \in y'' \setminus y' \ \& \ a \in y,$$

which contradicts (iv). Similarly the cases (ii) and (iii) are disjoint.

We conclude that the assignment of winning conditions is well-defined.

Then  $y$  is reachable for both winning strategies in  $(A, W)$  and winning strategies in  $(A, W)^\perp$ . Suppose  $\sigma$  is a winning strategy  $\sigma$  in  $(A, W)$ . By (iii) and Lemma 74,  $y$  is  $\sigma$ -reachable. From receptivity  $y_1$  is  $\sigma$ -reachable, say  $y_1 = \sigma x_1$  for some  $x_1 \in \mathcal{C}(S)$ . There is a  $+$ -maximal extension  $x'_1$  of  $x_1$  in  $\mathcal{C}^\infty(S)$ . By (i),  $\sigma x'_1$  cannot be a winning configuration. Hence there can be no winning strategy in  $(A, W)$ . In a dual fashion, there can be no winning strategy in  $(A, W)^\perp$ .  $\square$

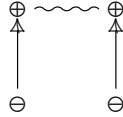
It is tempting to believe that a nondeterministic winning strategy always has a winning (weakly-)deterministic sub-strategy. However, this is not so, as the following examples show.

*Example 76.* A winning strategy need not have a winning deterministic sub-strategy. Consider the game  $(A, W)$  where  $A$  consists of two inconsistent events  $\ominus$  and  $\oplus$ , of the indicated polarity, and  $W = \{\{\ominus\}, \{\oplus\}\}$ . Consider the strategy  $\sigma$  in  $A$  given by the identity map  $\text{id}_A : a \rightarrow A$ . Then  $\sigma$  is a nondeterministic winning strategy—all  $+$ -maximal configurations in  $A$  are winning. However any sub-strategy must include  $\ominus$  by receptivity and cannot include  $\oplus$  if it is to be deterministic, whereupon it has  $\emptyset$  as a  $+$ -maximal configuration which is not winning.

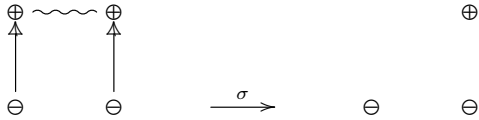
*Example 77.* Observe that the strategy  $\sigma$  of Example 76 is already weakly-deterministic—*cf.* Corollary 47. A winning strategy need not have a winning *weakly*-deterministic sub-strategy. Consider the game  $(A, W)$  where  $A$  consists of two  $-$ ve events 1, 2 and one  $+$ ve event 3 all consistent with each other and

$$W = \{\emptyset, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Let  $S$  be the event structure



and  $\sigma : S \rightarrow A$  the only possible total map of event structures with polarity:



Then  $\sigma$  is a winning strategy for which there is no weakly-deterministic sub-strategy.

## 8.6 Determinacy for well-founded games

**Definition 78.** A game  $A$  is well-founded if every configuration in  $\mathcal{C}^\infty(A)$  is finite.

It is shown that any well-founded concurrent game satisfying (‡) is determined.

### 8.6.1 Preliminaries

**Proposition 79.** *Let  $\mathcal{Q}$  be a family of finite partial orders closed under rigid inclusions, i.e. if  $q \in \mathcal{Q}$  and  $q' \hookrightarrow q$  is a rigid inclusion (regarded as a map of event structures) then  $q' \in \mathcal{Q}$ . The family  $\mathcal{Q}$  determines an event structure  $(P, \leq, \text{Con})$  as follows:*

- the events  $P$  are the prime partial orders in  $\mathcal{Q}$ , i.e. those finite partial orders in  $\mathcal{Q}$  with a top element;
- the causal dependency relation  $p' \leq p$  holds precisely when there is a rigid inclusion from  $p' \hookrightarrow p$ ;
- a finite subset  $X \subseteq P$  is consistent,  $X \in \text{Con}$ , iff there is  $q \in \mathcal{Q}$  and rigid inclusions  $p \hookrightarrow q$  for all  $p \in X$ . If  $x \in \mathcal{C}(P)$  then  $\bigcup x$ , the union of the partial orders in  $x$ , is in  $\mathcal{Q}$ . The function  $x \mapsto \bigcup x$  is an order-isomorphism from  $\mathcal{C}(P)$ , ordered by inclusion, to  $\mathcal{Q}$ , ordered by rigid inclusions.

Call a family of finite partial orders closed under rigid inclusions a *rigid family*. Observe:

**Proposition 80.** *Any stable family  $\mathcal{F}$  determines a rigid family: its configurations  $x$  possess a partial order  $\leq_x$  such that whenever  $x \subseteq y$  in  $\mathcal{F}$  there is a rigid inclusion  $(x, \leq_x) \hookrightarrow (y, \leq_y)$  between the corresponding partial orders.*

*Notation 81.* We shall use  $\text{Pr}(\mathcal{Q})$  for the construction described in Proposition 79. The construction extends that on stable families with the same name.

**Lemma 82.** *Let  $\sigma : S \rightarrow A$  be a strategy. Letting  $x, y \in \mathcal{C}(S)$ ,*

$$x^+ \subseteq y^+ \ \& \ \sigma x \subseteq \sigma y \implies x \subseteq y.$$

*Proof.* The proof relies on Proposition 41, characterising strategies. We first prove two special cases of the lemma.

*Special case  $\sigma x \subseteq^- \sigma y$ .* By assumption  $x^+ \subseteq y^+$ . Supposing  $s \in y^+ \setminus x^+$ , via the injectivity of  $\sigma$  on  $y$ , we obtain  $\sigma y \setminus \sigma x$  contains  $\sigma(s)$  a +ve event—a contradiction. Hence  $x^+ = y^+$ .

From Proposition 41(ii), as  $\sigma x \subseteq^- \sigma y$ , we obtain (a unique)  $x' \in \mathcal{C}(S)$  such that  $x \subseteq x'$  and  $\sigma x' = \sigma y$ :

$$\begin{array}{ccc} x & \cdots \subseteq & x' \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma x & \subseteq^- & \sigma y. \end{array}$$

Now  $[x^+] \sqsubseteq^- x$ , from which

$$\begin{array}{ccc} [x^+] & \sqsubseteq & x \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma[x^+] & \sqsubseteq^- & \sigma x. \end{array}$$

Combining the two diagrams:

$$\begin{array}{ccc} [x^+] & \sqsubseteq & x' \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma[x^+] & \sqsubseteq^- & \sigma y. \end{array}$$

As  $[y^+] \sqsubseteq^- y$ ,

$$\begin{array}{ccc} [y^+] & \sqsubseteq & y \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma[y^+] & \sqsubseteq^- & \sigma y. \end{array}$$

where, by Proposition 41(ii),  $y$  is the unique such configuration of  $S$ . But  $y^+ = x^+$  so this same property is shared by  $x'$ . Hence  $x' = y$  and  $x \sqsubseteq y$ .

Thus

$$x^+ \sqsubseteq y^+ \ \& \ \sigma x \sqsubseteq^- \sigma y \implies x \sqsubseteq y. \quad (1)$$

Note that, in particular,

$$x^+ = y^+ \ \& \ \sigma x = \sigma y \implies x = y. \quad (2)$$

*Special case*  $\sigma x \sqsubseteq^+ \sigma y$ . By Proposition 41(i), there is (a unique)  $y_1 \in \mathcal{C}(S)$  with  $y_1 \sqsubseteq y$  such that  $\sigma y_1 = \sigma x$ :

$$\begin{array}{ccc} y_1 & \cdots \sqsubseteq \cdots & y \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma x & \sqsubseteq^+ & \sigma y, \end{array}$$

Now  $x^+, y_1^+ \sqsubseteq y$  and  $\sigma x^+ = (\sigma x)^+ = \sigma y_1^+$ . So by the local injectivity of  $\sigma$  we obtain  $x^+ = y_1^+$ . By (2) above,  $x = y_1$ , whence  $x \sqsubseteq y$ . Thus

$$x^+ \sqsubseteq y^+ \ \& \ \sigma x \sqsubseteq^+ \sigma y \implies x \sqsubseteq y. \quad (3)$$

Any inclusion  $\sigma x \sqsubseteq \sigma y$  can be built as a composition of inclusions  $\sqsubseteq^-$  and  $\sqsubseteq^+$ , so the lemma follows from the special cases (1) and (3).  $\square$

**Lemma 83.** *Let  $\sigma : S \rightarrow A$  be a strategy for which no +ve event of  $S$  appears as a -ve event in  $A$ . Defining*

$$\mathcal{F}_\sigma =_{\text{def}} \{x^+ \cup (\sigma x)^- \mid x \in \mathcal{C}(S)\}$$

yields a stable family for which

$$\alpha_\sigma(s) = \begin{cases} s & \text{if } s \text{ is } +ve, \\ \sigma(s) & \text{if } s \text{ is } -ve. \end{cases}$$

is a map of stable families  $\alpha_\sigma : \mathcal{C}(S) \rightarrow \mathcal{F}_\sigma$  which induces an order-isomorphism

$$(\mathcal{C}(S), \subseteq) \cong (\mathcal{F}_\sigma, \subseteq)$$

taking  $x \in \mathcal{C}(S)$  to  $\alpha_\sigma x = x^+ \cup (\sigma x)^-$ . Defining

$$f_\sigma(e) = \begin{cases} \sigma(e) & \text{if } e \text{ is } +ve, \\ e & \text{if } e \text{ is } -ve \end{cases}$$

on events  $e$  of  $\mathcal{F}_\sigma$  yields a map of stable families  $f_\sigma : \mathcal{F}_\sigma \rightarrow \mathcal{C}(A)$  such that

$$\begin{array}{ccc} \mathcal{C}(S) & \xrightarrow{\alpha_\sigma} & \mathcal{F}_\sigma \\ & \searrow \sigma & \downarrow f_\sigma \\ & & \mathcal{C}(A) \end{array}$$

commutes.

*Proof.* A configuration  $x \in \mathcal{C}(S)$  has direct image

$$\alpha_\sigma x = x^+ \cup (\sigma x)^-$$

under the function  $\alpha_\sigma$ . Direct image under  $\alpha_\sigma$  is clearly surjective and preserves inclusions, and by Lemma 82 yields an order-isomorphism  $(\mathcal{C}(S), \subseteq) \cong (\mathcal{F}_\sigma, \subseteq)$ : if  $\alpha_\sigma x \subseteq \alpha_\sigma y$ , for  $x, y \in \mathcal{C}(S)$ , then  $x^+ \subseteq y^+$  and  $(\sigma x)^- \subseteq (\sigma y)^-$  by the disjointness of  $S^+$  and  $A$ , whence  $\sigma x \subseteq \sigma y$  so  $x \subseteq y$ .

It is now routine to check that  $\mathcal{F}_\sigma$  is a stable family and  $\alpha_\sigma$  is a map of stable families. For instance to show the stability property required of  $\mathcal{F}_\sigma$ , assume  $\alpha_\sigma x, \alpha_\sigma y \subseteq \alpha_\sigma z$ . Then  $x, y \subseteq z$  so  $\sigma x \cap y = (\sigma x) \cap (\sigma y)$  as  $\sigma$  is a map of event structures, and consequently  $(\sigma x \cap y)^- = (\sigma x)^- \cap (\sigma y)^-$ . Now reason

$$\begin{aligned} (\alpha_\sigma x) \cap (\alpha_\sigma y) &= (x^+ \cup (\sigma x)^-) \cap (y^+ \cup (\sigma y)^-) \\ &= (x^+ \cap y^+) \cup ((\sigma x)^- \cap (\sigma y)^-) \\ &\quad \text{—by distributivity with the disjointness of } S^+ \text{ and } A, \\ &= (x \cap y)^+ \cup (\sigma x \cap y)^- \\ &= (\alpha_\sigma x \cap y) \in \mathcal{F}_\sigma. \end{aligned}$$

From the definitions of  $\alpha_\sigma$  and  $f_\sigma$  it is clear that  $f_\sigma \alpha_\sigma(s) = \sigma(s)$  for all events of  $S$ . Any configuration of  $\mathcal{F}_\sigma$  is sent under  $f_\sigma$  to a configuration in  $\mathcal{C}(A)$  in a locally injective fashion, making  $f_\sigma$  a map of stable families; this follows from the matching properties of  $\sigma$ .  $\square$

## 8.7 Determinacy proof

**Definition 84.** Let  $A$  be an event structure with polarity. Let  $W \subseteq \mathcal{C}^\infty(A)$ . Let  $y \in \mathcal{C}^\infty(A)$ . Define  $A/y$  to be the event structure with polarity comprising events

$$\{a \in A \setminus y \mid y \cup [a]_A \in \mathcal{C}^\infty(A)\},$$

also called  $A/y$ , with consistency relation

$$X \in \text{Con}_{A/y} \text{ iff } X \subseteq_{\text{fin}} A/y \ \& \ y \cup [X]_A \in \mathcal{C}^\infty(A),$$

and causal dependency the restriction of that on  $A$ . Define  $W/y \subseteq \mathcal{C}^\infty(A/y)$  by

$$z \in W/y \text{ iff } z \in \mathcal{C}^\infty(A/y) \ \& \ y \cup z \in W.$$

Finally, define  $(A, W)/y =_{\text{def}} (A/y, W/y)$ .

**Proposition 85.** Let  $A$  be an event structure with polarity and  $y \in \mathcal{C}^\infty(A)$ . Then,

$$z \in \mathcal{C}^\infty(A/y) \text{ iff } z \subseteq A/y \ \& \ y \cup z \in \mathcal{C}^\infty(A).$$

Assume  $A$  is a *well-founded* event structure with polarity with winning conditions  $W \subseteq \mathcal{C}(A)$ . Assume the property  $(\ddagger)$  of  $A$ :

$$\forall y, y_1, y_2 \in \mathcal{C}(A). \ y \bar{\subset} y_1 \ \& \ y \bar{\subset}^+ y_2 \implies y_1 \uparrow y_2. \quad (\ddagger)$$

Observe that by repeated use of  $(\ddagger)$ , if  $x, y \in \mathcal{C}(A)$  with  $x \cap y \subseteq^+ x$  and  $x \cap y \subseteq^- y$ , then  $x \cup y \in \mathcal{C}(A)$ .

We show that the game  $(A, W)$  is determined. Assuming Player has no winning strategy we build a winning (counter) strategy for Opponent based on the following lemma.

**Lemma 86.** Assume game  $A$  is well-founded and satisfies  $(\ddagger)$ . Let  $W \subseteq \mathcal{C}(A)$ . Assume  $(A, W)$  has no winning strategy (for Player). Then,

$$\begin{aligned} & \forall x \in \mathcal{C}(A). \ \emptyset \subseteq^+ x \ \& \ x \in W \\ & \implies \\ & \exists y \in \mathcal{C}(A). \ x \subseteq^- y \ \& \ y \notin W \ \& \ (A, W)/y \text{ has no winning strategy.} \end{aligned}$$

*Proof.* Suppose otherwise, that under the assumption that  $(A, W)$  has no winning strategy, there is some  $x \in \mathcal{C}(A)$  such that

$$\begin{aligned} & \emptyset \subseteq^+ x \ \& \ x \in W \\ & \& \\ & \forall y \in \mathcal{C}(A). \ x \subseteq^- y \ \& \ y \notin W \implies (A, W)/y \text{ has a winning strategy.} \end{aligned}$$

We shall establish a contradiction by constructing a winning strategy for Player.

For each  $y \in \mathcal{C}(A)$  with  $x \subseteq^- y$  and  $y \notin W$ , choose a winning strategy

$$\sigma_y : S_y \rightarrow A/y.$$

By Lemma 83, we can replace  $\sigma_y$  by a stable family  $\mathcal{F}_y$  with all  $-ve$  events in  $A$  and a map of stable families  $f_y : \mathcal{F}_y \rightarrow \mathcal{C}(A)$ . It is easy to arrange that, within the collection of all such stable families,  $\mathcal{F}_{y_1}$  and  $\mathcal{F}_{y_2}$  are disjoint on  $+ve$  events whenever  $y_1$  and  $y_2$  are distinct. We build a putative stable family as

$$\begin{aligned} \mathcal{F} =_{\text{def}} & \{y \in \mathcal{C}(A) \mid \text{pol}_A(y \setminus x) \subseteq \{-\}\} \cup \\ & \{y \cup v \mid y \in \mathcal{C}(A) \ \& \ \text{pol}_A(y \setminus x) \subseteq \{-\} \ \& \ x \cup y \notin W \ \& \\ & \quad v \in \mathcal{F}_{x \cup y} \ \& \ + \in \text{pol } v \ \& \ y \cup f_{x \cup y} v \in \mathcal{C}(A)\}. \end{aligned}$$

[Note, in the second set-component, that  $x \cup y$  is a configuration by  $(\ddagger)$ .]

We assign events of  $\mathcal{F}$  the same polarities they have in  $A$  and the families  $\mathcal{F}_y$ .

We check that  $\mathcal{F}$  is indeed a stable family.

Clearly  $\emptyset \in \mathcal{F}$ . Assuming  $z_1, z_2 \subseteq z$  in  $\mathcal{F}$ , we require  $z_1 \cup z_2, z_1 \cap z_2 \in \mathcal{F}$ .

It is easily seen that if both  $z_1$  and  $z_2$  belong to the first set-component, so do their union and intersection. Suppose otherwise, without loss of generality, that  $z_2$  belongs to the second set-component. Then, necessarily,  $z$  is in the second set-component of  $\mathcal{F}$  and has the form  $z = y \cup v$  described there.

Consider the case where  $z_1 = y_1 \cup v_1$  and  $z_2 = y_2 \cup v_2$ , both belonging to the second set-component of  $\mathcal{F}$ . Then

$$x \cup y_1 = x \cup y_2 = x \cup y,$$

from the assumption that families  $\mathcal{F}_y$  are disjoint on  $+ve$  events for distinct  $y$ , and

$$v_1, v_2 \subseteq v \text{ in } \mathcal{F}_{x \cup y}.$$

It follows that  $x \cup (y_1 \cup y_2) = x \cup y \notin W$  and  $v_1 \cup v_2 \in \mathcal{F}_{x \cup y} = \mathcal{F}_{x \cup (y_1 \cup y_2)}$ . As  $z_1, z_2 \subseteq z$ ,

$$(y_1 \cup f_{x \cup y} v_1), (y_2 \cup f_{x \cup y} v_2) \subseteq (y \cup f_{x \cup y} v)$$

so

$$(y_1 \cup y_2) \cup f_{x \cup y} (v_1 \cup v_2) = (y_1 \cup f_{x \cup y} v_1) \cup (y_2 \cup f_{x \cup y} v_2) \in \mathcal{C}(A).$$

This ensures  $z_1 \cup z_2 = (y_1 \cup y_2) \cup (v_1 \cup v_2) \in \mathcal{F}$ . Similarly,  $x \cup (y_1 \cap y_2) = (x \cup y_1) \cap (x \cup y_2) = x \cup y \notin W$  and  $v_1 \cap v_2 \in \mathcal{F}_{x \cup y} = \mathcal{F}_{x \cup (y_1 \cap y_2)}$ . Checking

$$(y_1 \cap y_2) \cup f_{x \cup y} (v_1 \cap v_2) = (y_1 \cup f_{x \cup y} v_1) \cap (y_2 \cup f_{x \cup y} v_2) \in \mathcal{C}(A)$$

ensures  $z_1 \cap z_2 = (y_1 \cap y_2) \cup (v_1 \cap v_2) \in \mathcal{F}$ .

Consider the case where  $z_1 \in \mathcal{C}(A)$  belongs to the first and  $z_2 = y_2 \cup v_2$  to the second set-component of  $\mathcal{F}$ . As  $z_1 \subseteq y \cup v$  it has the form  $z_1 = y_1 \cup v_1$  where  $y_1 \in \mathcal{C}(A)$  with  $y_1 \subseteq y$  and  $v_1 \in \mathcal{F}_{x \cup y}$  with  $v_1 \subseteq v$ ; all the events of  $v_1 = z_1 \setminus (x \cup y)$  have  $-ve$  polarity which ensures  $v_1 \in \mathcal{F}_{x \cup y}$  by the receptivity of  $\sigma_y$ . Because  $v_2$  and  $v$  have  $+ve$  events in common,

$$x \cup y_2 = x \cup y,$$

while clearly

$$v_1, v_2 \subseteq v \text{ in } \mathcal{F}_{x \cup y}.$$

We deduce  $x \cup (y_1 \cup y_2) = x \cup y \notin W$  and  $v_1 \cup v_2 \in \mathcal{F}_{x \cup y} = \mathcal{F}_{x \cup (y_1 \cup y_2)}$  whence  $z_1 \cup z_2 = (y_1 \cup y_2) \cup (v_1 \cup v_2) \in \mathcal{F}$  after an easy check that  $(y_1 \cup y_2) \cup f_{x \cup y}(v_1 \cup v_2) \in \mathcal{C}(A)$ . We have  $y_2 \cup f_{x \cup y}v_2 \in \mathcal{C}(A)$ . But  $f_{x \cup y}$  is constant on  $-ve$  events so

$$z_1 \cap z_2 = z_1 \cap (y_2 \cup v_2) = z_1 \cap (y_2 \cup f_{x \cup y}v_2) \in \mathcal{C}(A),$$

and  $z_1 \cap z_2$  belongs to the first set-component of  $\mathcal{F}$ .

A routine check establishes that  $\mathcal{F}$  is coincidence-free, and uses that each family  $\mathcal{F}_y$  is coincidence-free when considering configurations of the second set-component.

Having established that  $\mathcal{F}$  is a stable family, we define a total map of stable families

$$f : \mathcal{F} \rightarrow \mathcal{C}(A)$$

by taking

$$f(e) = \begin{cases} e & \text{if } e \in x \text{ or } e \text{ is } -ve, \\ f_y(e) & \text{if } e \text{ is a } +ve \text{ event of } \mathcal{F}_y. \end{cases}$$

Defining  $\sigma$  to be the composite map of stable families

$$\mathcal{C}(\text{Pr}(\mathcal{F})) \xrightarrow{\text{max}} \mathcal{F} \xrightarrow{f} \mathcal{C}(A)$$

we also obtain a map of event structures

$$\sigma : \text{Pr}(\mathcal{F}) \rightarrow A$$

as the embedding of event structures in stable families is full and faithful. Ascribe to events  $p$  of  $\text{Pr}(\mathcal{F})$  the same polarities as events  $\text{max}(p)$  of  $\mathcal{F}$ . Clearly  $\sigma$  preserves polarities as  $f$  does, so  $\sigma$  is a total map of event structures with polarity. In fact,  $\sigma$  is a winning strategy for  $(A, W)$ .

To show receptivity of  $\sigma$  it suffices to show for all  $z \in \mathcal{F}$  that  $fz \bar{c} y'$  in  $\mathcal{C}(A)$  implies  $z \bar{c}' z'$  with  $\sigma z' = z$  for some unique  $z' \in \mathcal{F}$ . If  $z$  belongs to the first set-component of  $\mathcal{F}$  this is obvious—take  $z' = y'$ . Otherwise  $z$  belongs to the second set-component, and takes the form  $y \cup v$ , when receptivity follows from the receptivity of  $\sigma_{x \cup y}$ . No extra causal dependencies, over those of  $A$ , are introduced into  $y$  in the first set-component of  $\mathcal{F}$ . Considering  $y \cup v$  in the second set-component of  $\mathcal{F}$ , the only extra causal dependencies introduced in  $y \cup v$ , above those inherited from its image  $y \cup f_{x \cup y}v$  in  $A$ , are from  $v$  in  $\mathcal{F}_{x \cup y}$  and those making a  $+ve$  event of  $v$  in  $y \cup v$  depend on  $-ve$  events  $y \setminus x$ . For these reasons  $\sigma$  is also innocent, and a strategy in  $A$ .

To show  $\sigma$  is a winning strategy for  $(A, W)$  it suffices to show that  $fz \in W$  for every  $+ve$ -maximal configuration  $z \in \mathcal{F}$ . Let  $z$  be a  $+ve$ -maximal configuration of  $\mathcal{F}$ .

Suppose that  $z$  belongs to the first set-component of  $\mathcal{F}$  and, to obtain a contradiction, that  $fz \notin W$ . Then  $z = fz \in \mathcal{C}(A)$  and  $\text{pol } z \setminus x \subseteq \{-\}$ . By axiom ( $\ddagger$ ),  $x \uparrow y$ , so  $x \sqsubseteq z$  from the  $+ve$ -maximality of  $z$ . As  $x \sqsubseteq^- z$  and  $z \notin W$  the

strategy  $\sigma_z$  is winning in  $(A, W)/z$ . Because  $z$  is +-maximal in  $\mathcal{F}$  we must have  $\emptyset$  is +-maximal in  $\mathcal{F}_z$ . It follows that  $\emptyset \in W/z$ , i.e.  $z \in W$ —a contradiction.

Suppose that  $z$  belongs to the second set-component of  $\mathcal{F}$ , so that  $z$  has the form  $y \cup v$  with  $y \in \mathcal{C}(A)$  and  $v \in \mathcal{F}_{x \cup y}$ . By  $(\ddagger)$ ,  $x \subseteq y$ , as  $z$  is +-maximal in  $\mathcal{F}$ . Hence  $v \in \mathcal{F}_y$  and is necessarily +-maximal in  $\mathcal{F}_y$ , again from the +-maximality of  $z$ . As  $\sigma_y$  is winning,  $f_y v \in W/y$ . Therefore  $fz = y \cup f_y v \in W$ .

Finally, we have constructed a winning strategy  $\sigma$  in  $(A, W)$ —the contradiction required to establish the lemma.  $\square$

**Remark.** In the proof above we could instead build the strategy for Player, on which the proof by contradiction depends, out of a rigid family of finite partial orders. Recall that stable families, including configurations of event structures, are rigid families w.r.t. the order induced on configurations; finite configurations  $x$  determine finite partial orders  $(x, \leq_x)$ , which we call  $q(x)$  in the construction below. Define

$$\begin{aligned} \mathcal{Q} =_{\text{def}} & \{q(y) \mid y \in \mathcal{C}(A) \ \& \ \text{pol}_A(y \setminus x) \subseteq \{-\}\} \cup \\ & \{q(y); q(v) \mid y \in \mathcal{C}(A) \ \& \ \text{pol}_A(y \setminus x) \subseteq \{-\} \ \& \ x \cup y \notin W \ \& \\ & \quad v \in \mathcal{F}_{x \cup y} \ \& \ + \in \text{pol } v \ \& \ y \cup f_{x \cup y} v \in \mathcal{C}(A)\} \end{aligned}$$

where above  $q(y); q(v)$  is the least partial order on  $y \cup v$  in which events inherit causal dependencies from  $q(v)$ , from their images in  $q(y \cup f_{x \cup y} v)$  and in addition have the causal dependencies  $y^- \times v^+$ . The family  $\mathcal{Q}$  can be shown to be closed under rigid inclusions, and so a rigid family.  $\square$

**Theorem 87.** *Assume game  $A$  is well-founded, satisfies  $(\ddagger)$  and has winning conditions  $W \subseteq \mathcal{C}(A)$ . If  $(A, W)$  has no winning strategy for Player, then there is a winning (counter) strategy for Opponent.*

*Proof.* Assume  $(A, W)$  has no winning strategy for Player.

We build a winning counter-strategy for Opponent out of a rigid family of partial orders, themselves constructed from ‘alternating sequences’ of configurations of  $A$ .

Define an *alternating sequence* to be a sequence

$$x_1, y_1, x_2, y_2, \dots, x_i, y_i, \dots, x_k, y_k, x_{k+1}$$

of length  $k + 1 \geq 1$  of configurations of  $A$  such that

$$\emptyset \leq^+ x_1 \leq^- y_1 \leq^+ x_2 \leq^- y_2 \leq^- \dots \leq^+ x_i \leq^- y_i \leq^+ \dots \leq^+ x_k \leq^- y_k \leq^+ x_{k+1}$$

with

$$x_i \in W \ \& \ y_i \notin W \ \& \ (A, W)/y_i \text{ has no winning strategy,}$$

when  $1 \leq i \leq k$ . It is important that  $x_{k+1}$ , which may be  $\emptyset$ , need not be in  $W$ . In particular, we allow the alternating singleton sequence  $x_1$  comprising a single configuration of  $A$  with  $\emptyset \leq^+ x_1$  without necessarily having  $x_1 \in W$ .

For each alternating sequence  $x_1, y_1, \dots, x_k, y_k, x_{k+1}$  define the partial order  $Q(x_1, y_1, \dots, x_k, y_k, x_{k+1})$  to comprise the partial order on  $x_{k+1}$  inherited from  $A$  together with additional causal dependencies given by the pairs in

$$x_i^+ \times (y_i \setminus x_i), \text{ where } 1 \leq i \leq k.$$

We define  $\mathcal{Q}$  to be the rigid family comprising the set of all partial orders got from alternating sequences, closed under rigid inclusions.

Form the event structure  $\text{Pr}(\mathcal{Q})$  as described in Proposition 79. Assign the same polarity to an event in  $\text{Pr}(\mathcal{Q})$  as its top event in  $A$ . Recall from Proposition 79 the order-isomorphism  $\mathcal{C}(\text{Pr}(\mathcal{Q})) \cong \mathcal{Q}$  given by  $x \mapsto \bigcup x$  for  $x \in \mathcal{C}(\text{Pr}(\mathcal{Q}))$ . The map

$$\tau : \text{Pr}(\mathcal{Q}) \rightarrow A$$

taking  $p \in \text{Pr}(\mathcal{Q})$  to its top event is a total map of event structures with polarity. Writing  $T : \mathcal{Q} \rightarrow \mathcal{C}(A)$  for the function taking  $q \in \mathcal{Q}$  to its set of underlying events,  $\tau x = T(\bigcup x)$  for all  $x \in \mathcal{C}(\text{Pr}(\mathcal{Q}))$ , *i.e.* the diagram

$$\begin{array}{ccc} \mathcal{C}(\text{Pr}(\mathcal{Q})) & \cong & \mathcal{Q} \\ & \searrow \tau & \downarrow T \\ & & \mathcal{C}(A) \end{array}$$

commutes. We shall reason about order-properties of  $\tau$  via the function  $T$ .

We claim that  $\tau$  is a winning counter-strategy, in other words a winning strategy for Opponent, in which the roles of  $+$  and  $-$  are reversed.

Because the construction of the partial orders in  $\mathcal{Q}$  only introduces extra causal dependencies of  $-$ ve events on  $+$ ve events,  $\tau$  is innocent (remember the reversal of polarities). To check receptivity of  $\tau$  it suffices to show that for  $q \in \mathcal{Q}$  assuming  $T(q) \stackrel{a}{-c} z'$  in  $\mathcal{C}(A)$ , where  $\text{pol}_A(a) = +$ , there is a unique  $q' \in \mathcal{Q}$  such that  $q \stackrel{a}{-c} q'$  and  $T(q') = z'$ . Any such extension  $q'$  must comprise the partial order  $q$  extended by the event  $a$ . As  $a$  is  $+$ ve the events on which it immediately depends in  $q'$  will coincide with those on which  $a$  immediately depends in  $z'$ , guaranteeing the uniqueness of  $q'$ . It remains to show the existence of  $q'$ .

By assumption,  $q$  rigidly embeds in  $Q(x_1, y_1, \dots, x_k, y_k, x_{k+1})$  for some alternating sequence  $x_1, y_1, \dots, x_k, y_k, x_{k+1}$ . In the case where  $q$  consists of purely  $+$ ve events, take  $q' =_{\text{def}} Q(z')$ . Otherwise, consider the largest  $i$  for which  $T(q) \cap (y_i \setminus x_i) \neq \emptyset$ . Then,

$$\text{pol}_A T(q) \setminus y_i \subseteq \{+\}. \quad (1)$$

From the construction of  $Q(x_1, y_1, \dots, x_k, y_k, x_{k+1})$  and the rigidity of the inclusion of  $q$  in  $Q(x_1, y_1, \dots, x_k, y_k, x_{k+1})$  we obtain

$$x_i^+ \subseteq T(q). \quad (2)$$

From (2),  $T(q) \subseteq^- T(q) \cup y_i$  and, by assumption,  $T(q) \stackrel{a}{-c} z'$  with  $\text{pol}_A(a) = +$ . Using (†), their union remains in  $\mathcal{C}(A)$ , and we can define

$$x' =_{\text{def}} T(q) \cup y_i \cup \{a\} \in \mathcal{C}(A).$$

Note that

$$x_1, y_1, \dots, x_i, y_i, x'$$

is an alternating sequence because  $y_i \subseteq^+ x'$  by (1) and it is built from an alternating sequence  $x_1, y_1, \dots, x_k, y_k, x_{k+1}$ . Restricting  $Q(x_1, y_1, \dots, x_i, y_i, x')$  to events  $z$  we obtain a partial order  $q'$  for which  $q \text{---} c q'$  in  $\mathcal{Q}$  and  $T(q') = z$ .

We now show that  $\tau$  is winning for Opponent. For this it suffices to show that if  $q \in \mathcal{Q}$  is --maximal then  $T(q) \notin W$ . Assume  $q \in \mathcal{Q}$  is --maximal in  $\mathcal{Q}$ . Necessarily  $q$  embeds rigidly in  $Q(x_1, y_1, \dots, x_k, y_k, x_{k+1})$  for some alternating sequence  $x_1, y_1, \dots, x_k, y_k, x_{k+1}$ .

In the case where  $q$  consists of purely +ve events

$$\emptyset \subseteq^+ T(q) \text{ in } \mathcal{C}(A).$$

Suppose  $T(q) \in W$ . By Lemma 86, for some  $y \in \mathcal{C}(A)$ ,

$$T(q) \subseteq^- y \ \& \ y \notin W.$$

But then there is a strict extension  $q \hookrightarrow Q(T(q), y, \emptyset)$  of  $q$  by -ve events in  $\mathcal{Q}$ , and  $q$  is not --maximal—a contradiction.

In the case where  $q$  has -ve events, we may take the largest  $i$  for which  $T(q) \cap (y_i \setminus x_i) \neq \emptyset$ . As earlier,

$$(1) \text{ pol}_A T(q) \setminus y_i \subseteq \{+\} \quad \& \quad (2) x_i^+ \subseteq T(q).$$

As  $q$  is --maximal,  $y_i \subseteq T(q)$ , whence by (1),

$$y_i \subseteq^+ T(q).$$

Suppose, to obtain a contradiction, that  $T(q) \in W$ . The game  $(A, W)/y_i$  has no winning strategy. By Lemma 86, given

$$\emptyset \subseteq^+ x \text{ =}_{\text{def}} T(q) \setminus y_i$$

in  $\mathcal{C}((A, W)/y_i)$  there is  $y \in \mathcal{C}((A, W)/y_i)$  with

$$x \subseteq^- y \ \& \ y \notin W/y_i.$$

Let  $x'_{i+1} \text{ =}_{\text{def}} T(q)$  and  $y'_{i+1} \text{ =}_{\text{def}} y_i \cup y \notin W$ . Then,

$$x_1, y_1, \dots, x_i, y_i, x'_{i+1}, y'_{i+1}, \emptyset$$

is an alternating sequence which strictly extends  $q$  by -ve events, contradicting its --maximality.

We conclude that  $\tau$  is a winning strategy for Opponent.  $\square$

**Corollary 88.** *If a well-founded game  $A$  satisfies ( $\ddagger$ ) then  $(A, W)$  is determined for any winning conditions  $W$ .*

## 8.8 Satisfaction in the predicate calculus

The syntax for predicate calculus: formulae are given by

$$\phi, \psi, \dots ::= R(x_1, \dots, x_k) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \exists x. \phi \mid \forall x. \phi$$

where  $R$  ranges over basic relation symbols of a fixed arity and  $x, x_1, x_2, \dots, x_k$  over variables.

A model  $M$  for the predicate calculus comprises a non-empty universe of values  $V_M$  and an interpretation for each of the relation symbols as a relation of appropriate arity on  $V_M$ . Following Tarski we can then define by structural induction the truth of a formula of predicate logic w.r.t. an assignment of values in  $V_M$  to the variables of the formula. We write

$$\rho \models_M \phi$$

iff formula  $\phi$  is true in  $M$  w.r.t. environment  $\rho$ ; we take an environment to be a function from variables to values.

W.r.t. a model  $M$  and an environment  $\rho$ , we can denote a formula  $\phi$  by  $\llbracket \phi \rrbracket_{M\rho}$ , a concurrent game with winning conditions, so that  $\rho \models_M \phi$  iff the game  $\llbracket \phi \rrbracket_{M\rho}$  has a winning strategy.

The denotation as a game is defined by structural induction:

$$\begin{aligned} \llbracket R(x_1, \dots, x_k) \rrbracket_{M\rho} &= \begin{cases} (\emptyset, \{\emptyset\}) & \text{if } \rho \models_M R(x_1, \dots, x_k), \\ (\emptyset, \emptyset) & \text{otherwise.} \end{cases} \\ \llbracket \phi \wedge \psi \rrbracket_{M\rho} &= \llbracket \phi \rrbracket_{M\rho} \otimes \llbracket \psi \rrbracket_{M\rho} \\ \llbracket \phi \vee \psi \rrbracket_{M\rho} &= \llbracket \phi \rrbracket_{M\rho} \wp \llbracket \psi \rrbracket_{M\rho} \\ \llbracket \neg\phi \rrbracket_{M\rho} &= (\llbracket \phi \rrbracket_{M\rho})^\perp \\ \llbracket \exists x. \phi \rrbracket_{M\rho} &= \bigoplus_{v \in V_M} \llbracket \phi \rrbracket_{M\rho[v/x]} \\ \llbracket \forall x. \phi \rrbracket_{M\rho} &= \bigotimes_{v \in V_M} \llbracket \phi \rrbracket_{M\rho[v/x]}. \end{aligned}$$

We use  $\rho[v/x]$  to mean the environment  $\rho$  updated to assign value  $v$  to variable  $x$ . The game  $(\emptyset, \{\emptyset\})$  the unit w.r.t.  $\otimes$  is the game used to denote true and the game  $(\emptyset, \{\emptyset\})$  the unit w.r.t.  $\wp$  to denote false. Denotations of conjunctions and disjunctions are denoted by the operations of  $\otimes$  and  $\wp$  on games, while negations denote dual games. Universal and existential quantifiers denote *prefixed sums* of games, operations which we now describe.

The prefixed game  $\oplus.(A, W)$  comprises the event structure with polarity  $\oplus.A$  in which all the events of  $A$  are made to causally depend on a fresh +ve event  $\oplus$ . Its winning conditions are those configurations  $x \in \mathcal{C}^\infty(\oplus.A)$  of the form  $\{\oplus\} \cup y$  for some  $y \in W$ . The game  $\bigoplus_{v \in V} (A_v, W_v)$  has underlying event structure with polarity the sum (=coproduct)  $\sum_{v \in V} \oplus.A_v$  with a configuration winning iff it is the image of a winning configuration in a component under the injection to the sum. Note in particular that the empty configuration of  $\bigoplus_{v \in V} G_v$  is not

winning—Player must make a move in order to win. The game  $\ominus_{v \in V} G_v$  is defined dually, as  $(\oplus_{v \in V} G_v^+)^+$ . In this game the empty configuration is winning but Opponent gets to make the first move. More explicitly, the prefixed game  $\ominus.(A, W)$  comprises the event structure with polarity  $\ominus.A$  in which all the events of  $A$  are made to causally depend on the previous occurrence of an opponent event  $\ominus$ , with winning configurations of the form  $\{\ominus\} \cup y$  where  $y \in W$ . Writing  $G_v = (A_v, W_v)$ , the underlying event structure of  $\ominus_{v \in V} G_v$  is the sum  $\sum_{v \in V} \ominus.A_v$  with a configuration winning iff it is empty or the image under injection of a winning configuration in a prefixed component.

It is easy to check by structural induction that:

**Proposition 89.** *For any formula  $\phi$  the game  $\llbracket \phi \rrbracket_{M\rho}$  is well-founded and race-free (i.e. satisfies Axiom  $(\ddagger)$ ), so a determined game by the result of the last section.*

The following facts are useful for building strategies.

**Proposition 90.**

- (i) *If  $\sigma : S \rightarrow A$  is a strategy in  $A$  and  $\tau : T \rightarrow B$  is a strategy in  $B$ , then  $\sigma \parallel \tau : S \parallel T \rightarrow A \parallel B$  is a strategy in  $A \parallel B$ .*
- (ii) *If  $\sigma : S \rightarrow T$  is a strategy in  $T$  and  $\tau : T \rightarrow B$  is a strategy in  $B$ , then their composition as maps of event structures with polarity  $\tau \sigma : S \rightarrow B$  is a strategy in  $B$ .*

*Proof.* It is easy to check that the properties of receptivity and innocence are preserved by parallel composition and composition of maps.  $\square$

There are ‘projection’ strategies from a tensor product of games to its components:

**Proposition 91.** *Let  $G = (A, W_G)$  and  $H = (B, W_H)$  be race-free games with winning conditions. The map of event structures with polarity*

$$\text{id}_{A^+} \parallel \gamma_B : A^+ \parallel \mathbb{C}_B \rightarrow A^+ \parallel B^+ \parallel B$$

*is a winning strategy  $p_H : G \otimes H \rightarrow H$ . The map of event structures with polarity*

$$\text{id}_{B^+} \parallel \gamma_A : B^+ \parallel \mathbb{C}_A \rightarrow B^+ \parallel A^+ \parallel A \cong A^+ \parallel B^+ \parallel A$$

*is a winning strategy  $p_G : G \otimes H \rightarrow G$ .*

*Proof.* By Proposition 90, as  $\text{id}_{A^+}$  is a strategy in  $A^+$  and  $\gamma_B$  is a strategy in  $B^+ \parallel B$  the map  $p_H = \text{id}_{A^+} \parallel \gamma_B$  is certainly a strategy in  $A^+ \parallel B^+ \parallel B$ .

We need to check that  $p_H$  is a winning strategy in  $G \otimes H \rightarrow H$ . Consider  $x$ , a +-maximal configuration of  $A^+ \parallel \mathbb{C}_B$ . As  $B$  is race-free, the copy-cat strategy  $\gamma_B$  is winning in  $H \rightarrow H$ . Consequently if  $x$  images to a winning configuration in  $G \otimes H$  on the left of  $G \otimes H \rightarrow H$  it will image to a winning configuration in  $H$  on the right of  $G \otimes H \rightarrow H$ . (Recall a winning configuration of  $G \otimes H$  is essentially

the union of a winning configuration in  $G$  together with a winning configuration in  $H$ .) Consequently,  $x$  images to a winning configuration in  $G \otimes H \rightarrow H$ , as is required for  $p_H$  to be a winning strategy.

The strategy  $p_G$  is defined analogously but for the isomorphism  $B^\perp \| A^\perp \| A \cong A^\perp \| B^\perp \| A$  which does not disturb its winning nature.  $\square$

The following lemma is used to build and deconstruct strategies in prefixed sums of games. The lemma concerns the more basic prefixed sums of event structures. These are built as coproducts  $\sum_{i \in I} \bullet.B_i$  of event structures  $\bullet.B_i$  in which an event  $\bullet$  is prefixed to  $B_i$ , making all the events in  $B_i$  causally depend on  $\bullet$ .

**Lemma 92.** *Suppose  $f : A \rightarrow \sum_{i \in I} \bullet.B_i$  is a total map of event structures, with codomain a prefixed sum. Then,  $A$  is isomorphic to an prefixed sum,  $A \cong \sum_{j \in J} \bullet.A_j$ , and there is a function  $r : J \rightarrow I$  and total maps of event structures  $f_j : A_j \rightarrow B_{r(j)}$  for which*

$$\begin{array}{ccc} \sum_{j \in J} \bullet.A_j & \cong & A \\ \downarrow [\bullet.f_j]_{j \in J} & \swarrow f & \\ \sum_{i \in I} \bullet.B_i & & \end{array}$$

commutes.

*Proof.* Let  $J$  be the subset of events of  $A$  whose images are prefix events  $\bullet$  in  $\sum_{i \in I} \bullet.B_i$ . As  $f$  is a map of event structures any distinct pairs of events in  $J$  are inconsistent. Moreover, every event of  $A$  is  $\leq_A$ -above a necessarily unique event in  $J$ . It follows that the events of  $J$  are  $\leq_A$ -minimal with  $A \cong \sum_{j \in J} \bullet.A_j$ ; the event structure  $A_j$  is  $A/\{j\}$ , that part of the event structure strictly above the event  $j$ . Each event  $j \in J$  is sent to a unique prefix event  $f(j)$  in  $\sum_{i \in I} \bullet.B_i$ . Thus  $f$  determines a function  $r : J \rightarrow I$  and maps  $f_j : A_j \rightarrow B_{r(j)}$  for all  $j \in J$ . By construction the map  $f$  is reassembled, up to isomorphism, as the unique mediating map  $[\bullet.f_j]_{j \in J}$  for which

$$\begin{array}{ccccc} \bullet.A_j & \xrightarrow{\text{in}_j^A} & \sum_{j \in J} \bullet.A_j & \cong & A \\ \bullet.f_j \downarrow & & \downarrow [\bullet.f_j]_{j \in J} & \swarrow f & \\ \bullet.B_{r(j)} & \xrightarrow{\text{in}_{r(j)}^B} & \sum_{i \in I} \bullet.B_i & & \end{array}$$

commutes for all  $j \in J$ .  $\square$

**Lemma 93.** *Let  $G, H, G_v$ , where  $v \in V$ , be race-free games with winning conditions. Then,*

- (i)  $G \otimes H$  has a winning strategy iff  $G$  has a winning strategy and  $H$  has a winning strategy.

(ii)  $\bigoplus_{v \in V} G_v$  has a winning strategy iff  $G_v$  has a winning strategy for some  $v \in V$ .

(iii)  $\bigotimes_{v \in V} G_v$  has a winning strategy iff  $G_v$  has a winning strategy for all  $v \in V$ .

If in addition  $G$  and  $H$  are determined,

(iv)  $G \wp H$  has a winning strategy iff  $G$  has a winning strategy or  $H$  has a winning strategy.

*Proof.* Throughout write  $G_v = (A_v, W_v)$ , where  $v \in V$ .

(i) ‘Only if’: If  $G \otimes H$  has a winning strategy  $\sigma : (\emptyset, \{\emptyset\}) \dashrightarrow G \otimes H$ , then the compositions  $p_G \circ \sigma$  and  $p_H \circ \sigma$  provide winning strategies in  $G$  and  $H$ , respectively. ‘If’: If  $G = (A, W_G)$  and  $H = (B, W_H)$  have winning strategies given as maps of event structures with polarity  $\sigma : S \rightarrow A$  and  $\tau : T \rightarrow B$  then the map  $\sigma \parallel \tau : S \parallel T \rightarrow A \parallel B$  is a winning strategy in  $G \otimes H$ .

(ii) ‘Only if’: Suppose  $\sigma : S \rightarrow \sum_{v \in V} \oplus .A_v$  is a winning strategy in  $\bigoplus_{v \in V} G_v$ . As  $\emptyset$  is not winning in the game,  $S$  must be nonempty. By Lemma 92,  $S$  decomposes into a prefixed sum necessarily nonempty and of the form  $\sum_{j \in J} \oplus .S_j$  with maps, now necessarily total maps of event structures with polarity,  $\sigma_j : S_j \rightarrow A_{v(j)}$ . Because  $\sigma$  is winning any such map will be a winning strategy in  $G_{v(j)}$ . ‘If’: Suppose  $\sigma_v : S_v \rightarrow A_v$  is a winning strategy in  $G_v$ . Prefixing we obtain  $\oplus .\sigma_v : \oplus .S_v \rightarrow \oplus .A_v$ , a winning strategy in  $\oplus .G_v$ . Composing with the winning ‘injection’ strategy  $in_v : \oplus .G_v \dashrightarrow \sum_{v \in V} \oplus .G_v$  defined below we obtain a winning strategy in  $\bigoplus_{v \in V} G_v$ . The injection strategy is built from the injection map of event structures with polarity

$$in_v : \oplus .A_v \rightarrow \sum_{v \in V} \oplus .A_v .$$

as the composite map

$$In_v : \mathbb{C}_{\oplus .A_v} \xrightarrow{\gamma_{\oplus .A_v}} (\oplus .A_v)^\perp \parallel \oplus .A_v \xrightarrow{\text{id}_{(\oplus .A_v)^\perp} \parallel in_v} (\oplus .A_v)^\perp \parallel \sum_{v \in V} \oplus .A_v .$$

Proposition 90 is used to show  $In_v$  is a strategy. It can be seen that  $in_v$  is both receptive and innocent so a strategy in  $\sum_{v \in V} \oplus .A_v$ . The map  $\text{id}_{(\oplus .A_v)^\perp}$  is a strategy. Hence  $\text{id}_{(\oplus .A_v)^\perp} \parallel in_v$  is a strategy. As the composition of two strategy maps,  $In_v$  is a strategy in  $(\oplus .A_v)^\perp \parallel \sum_{v \in V} \oplus .A_v$ . It is a winning strategy because, as is easily seen from the explicit composite form of  $In_v$ , the image under  $In_v$  of a  $+$ -maximal configuration in  $\mathbb{C}_{\oplus .A_v}$  is winning.

(iii) ‘Only if’: Defining  $P_v =_{\text{def}} In_v^\perp$ , where  $In_v : \oplus .G_v \dashrightarrow \bigoplus_{v \in V} G_v^\perp$  is an instance of an injection strategy defined above, we obtain by duality a winning strategy

$$P_v : \bigotimes_{v \in V} G_v \dashrightarrow \bigotimes .G_v ,$$

for any  $v \in V$ . Let  $v \in V$ . By composition with  $P_v$  a winning strategy in  $\Theta_{v \in V} G_v$  yields a winning strategy in the component  $\Theta.G_v$ . By Lemma 92 in a strategy  $\sigma : S \rightarrow \Theta.A_v$  the event structure  $S$  decomposes into a prefixed sum, where the prefixing events are necessarily all -ve. As  $\sigma$  is receptive the sum must be a unary prefixed sum of the form  $\Theta.S'$ . Lemma 92 provides a map  $\sigma' : S' \rightarrow A_v$ . From  $\sigma$  being winning the map  $\sigma'$  will be a winning strategy in  $G_v$ . *'If'*: Suppose  $\sigma_v : S_v \rightarrow A_v$  is a winning strategy in  $G_v$ , for all  $v \in V$ . Prefixing we obtain winning strategies  $\Theta.\sigma_v : \Theta.S_v \rightarrow \Theta.A_v$  in  $\Theta.G_v$ . Forming the sum  $\sum_{v \in V} \Theta.\sigma_v : \sum_{v \in V} \Theta.S_v \rightarrow \Theta.\sigma_v : \sum_{v \in V} \Theta.A_v$  we obtain a strategy winning in  $\Theta_{v \in V} G_v$ .

(iv) Now suppose  $G$  and  $H$  are determined. *'If'*: The dual winning strategies  $p_{G^\perp}^\perp : G \dashv\rightarrow G \wp H$  and  $p_{H^\perp}^\perp : H \dashv\rightarrow G \wp H$  compose with a winning strategy  $(\emptyset, \{\emptyset\}) \dashv\rightarrow G$ , or respectively a winning strategy  $(\emptyset, \{\emptyset\}) \dashv\rightarrow H$ , to yield a winning strategy  $(\emptyset, \{\emptyset\}) \dashv\rightarrow G \wp H$ . *'Only if'*: Suppose  $G \wp H$  has a winning strategy. Then  $G^\perp \otimes H^\perp = (G \wp H)^\perp$  has no winning strategy. Hence by (i),  $G^\perp$  has no winning strategy or  $H^\perp$  has no winning strategy. From determinacy,  $G$  has a winning strategy or  $H$  has a winning strategy.  $\square$

**Theorem 94.** *For all predicate-calculus formulae  $\phi$  and environments  $\rho$ ,  $\rho \models_M \phi$  iff the game  $\llbracket \phi \rrbracket_{M\rho}$  has a winning strategy.*

*Proof.* By Proposition 89 the games  $\llbracket \phi \rrbracket_{M\rho}$  obtained from formulae  $\phi$  are race-free and determined. The proof is by structural induction on  $\phi$ .

The base case where  $\phi$  is  $R(x_1, \dots, x_k)$  is obvious; the game  $(\emptyset, \{\emptyset\})$  has as (unique) winning strategy the map  $\emptyset \rightarrow \emptyset$ , while  $(\emptyset, \emptyset)$  has no winning strategy.

For the case  $\phi \wedge \psi$ , reason

$$\begin{aligned} \rho \models_M \phi \wedge \psi &\iff \rho \models_M \phi \ \& \ \rho \models_M \psi \\ &\iff \llbracket \phi \rrbracket_{M\rho} \text{ has a winning strategy} \ \& \ \llbracket \psi \rrbracket_{M\rho} \text{ has a winning strategy, by induction,} \\ &\iff \llbracket \phi \rrbracket_{M\rho} \otimes \llbracket \psi \rrbracket_{M\rho} \text{ has a winning strategy, by Lemma 93(i),} \\ &\iff \llbracket \phi \wedge \psi \rrbracket_{M\rho} \text{ has a winning strategy.} \end{aligned}$$

In the case  $\phi \vee \psi$ ,

$$\begin{aligned} \rho \models_M \phi \vee \psi &\iff \rho \models_M \phi \ \text{or} \ \rho \models_M \psi \\ &\iff \llbracket \phi \rrbracket_{M\rho} \text{ has a winning strategy or} \ \llbracket \psi \rrbracket_{M\rho} \text{ has a winning strategy, by induction,} \\ &\iff \llbracket \phi \rrbracket_{M\rho} \wp \llbracket \psi \rrbracket_{M\rho} \text{ has a winning strategy, by Lemma 93(iv),} \\ &\iff \llbracket \phi \vee \psi \rrbracket_{M\rho} \text{ has a winning strategy.} \end{aligned}$$

In the case  $\neg\phi$ ,

$$\begin{aligned} \rho \models_M \neg\phi &\iff \rho \not\models_M \phi \\ &\iff \llbracket \phi \rrbracket_{M\rho} \text{ has no winning strategy, by induction,} \\ &\iff (\llbracket \phi \rrbracket_{M\rho})^\perp \text{ has a winning strategy, by determinacy.} \end{aligned}$$

In the case  $\exists x. \phi$ ,

$$\begin{aligned}
\rho \models_M \exists x. \phi &\iff \rho[v/x] \models_M \phi \text{ for some } v \in V \\
&\iff \llbracket \phi \rrbracket_M \rho[v/x] \text{ has a winning strategy, for some } v \in V, \text{ by induction,} \\
&\iff \bigoplus_{v \in V} \llbracket \phi \rrbracket_M \rho[v/x] \text{ has a winning strategy, by Lemma 93(ii),} \\
&\iff \llbracket \exists x. \phi \rrbracket_M \rho \text{ has a winning strategy.}
\end{aligned}$$

In the case  $\forall x. \phi$ ,

$$\begin{aligned}
\rho \models_M \forall x. \phi &\iff \rho[v/x] \models_M \phi \text{ for all } v \in V \\
&\iff \llbracket \phi \rrbracket_M \rho[v/x] \text{ has a winning strategy, for all } v \in V, \text{ by induction,} \\
&\iff \bigotimes_{v \in V} \llbracket \phi \rrbracket_M \rho[v/x] \text{ has a winning strategy, by Lemma 93(iii),} \\
&\iff \llbracket \forall x. \phi \rrbracket_M \rho \text{ has a winning strategy.}
\end{aligned}$$

□

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# Bibliography

- [1] Nielsen, M., Plotkin, G., Winskel, G.: Petri nets, event structures and domains. *Theoretical Computer Science* **13** (1981) 85–108
- [2] Winskel, G., Nielsen, M.: Models for concurrency. In Abramsky, S., Gabbay, D., eds.: *Semantics and Logics of Computation*. OUP (1995)
- [3] Saunders-Evans, L., Winskel, G.: Event structure spans for nondeterministic dataflow. *Electr. Notes Theor. Comput. Sci.* 175(3): 109-129 (2007)
- [4] Winskel, G.: Event structure semantics for CCS and related languages. In: *ICALP'82*. Volume 140 of LNCS., Springer (1982)
- [5] Rideau, S., Winskel, G.: Concurrent strategies. In: *LICS 2011*, IEEE Computer Society (2011)
- [6] Joyal, A.: Remarques sur la théorie des jeux à deux personnes. *Gazette des sciences mathématiques du Québec*, 1(4) (1997)
- [7] Winskel, G.: Event structures with symmetry. *Electr. Notes Theor. Comput. Sci.* 172: 611-652 (2007)
- [8] Laird, J.: A games semantics of idealized CSP. Vol 45 of *Electronic Books in Theor. Comput. Sci.* (2001)
- [9] Ghica, D.R., Murawski, A.S.: Angelic semantics of fine-grained concurrency. In: *FOSSACS'04*, LNCS 2987, Springer (2004)
- [10] Melliès, P.A., Mimram, S.: Asynchronous games : innocence without alternation. In: *CONCUR '07*. Volume 4703 of LNCS., Springer (2007)
- [11] Katovsky, A.: Concurrent games. First-year report for PhD study, Computer Lab, Cambridge (2011)
- [12] Curien, P.L.: On the symmetry of sequentiality. In: *MFPS*. Number 802 in LNCS, Springer (1994) 29–71
- [13] Hyland, M.: Game semantics. In Pitts, A., Dybjer, P., eds.: *Semantics and Logics of Computation*. Publications of the Newton Institute (1997)

- [14] Harmer, R., Hyland, M., Melliès, P.A.: Categorical combinatorics for innocent strategies. In: LICS '07, IEEE Computer Society (2007)
- [15] Melliès, P.A.: Asynchronous games 2: The true concurrency of innocence. *Theor. Comput. Sci.* 358(2-3): 200-228 (2006)
- [16] Abramsky, S., Melliès, P.A.: Concurrent games and full completeness. In: LICS '99, IEEE Computer Society (1999)
- [17] Hyland, J.M.E., Ong, C.H.L.: On full abstraction for PCF: I, II, and III. *Inf. Comput.* 163(2): 285-408 (2000)
- [18] Abramsky, S., Jagadeesan, R., Malacaria, P.: Full abstraction for PCF. *Inf. Comput.* 163(2): 409-470 (2000)
- [19] Varacca, D., Völzer, H., Winskel, G.: Probabilistic event structures and domains. *Theor. Comput. Sci.* 358(2-3): 173-199 (2006)
- [20] Hyland, M.: Some reasons for generalising domain theory. *Mathematical Structures in Computer Science* 20(2) (2010) 239–265
- [21] Cattani, G.L., Winskel, G.: Profunctors, open maps and bisimulation. *Mathematical Structures in Computer Science* 15(3) (2005) 553–614
- [22] Curien, P.L., Plotkin, G.D., Winskel, G.: Bistructures, bidomains, and linear logic. In: *Proof, Language, and Interaction, essays in honour of Robin Milner*, MIT Press (2000) 21–54
- [23] Abramsky, S.: Semantics of interaction. In Pitts, A., Dybjer, P., eds.: *Semantics and Logics of Computation*. Publications of the Newton Institute (1997)

# Appendix A

## Exercises

### On event structures and stable families

*Recommended exercises: 1, 3, 4, 5 (Harder), 6, 7, 10.*

*Exercise 1.* Let  $(A, \leq_A, \text{Con}_A), (B, \leq_B, \text{Con}_B)$  be event structures. Let  $f : A \rightarrow B$ . Show  $f$  is a map of event structures,  $f : (A, \leq_A, \text{Con}_A) \rightarrow (B, \leq_B, \text{Con}_B)$ , iff

(i)  $\forall a \in A, b \in B. b \leq_B f(a) \implies \exists a' \in A. a' \leq_A a \ \& \ f(a') = b$ , and

(ii)  $\forall X \in \text{Con}_A. fX \in \text{Con}_B \ \& \ \forall a_1, a_2 \in X. f(a_1) = f(a_2) \implies a_1 = a_2$ .

□

*Exercise 2.* Show a map  $f : A \rightarrow B$  of event structures is mono iff the function  $\mathcal{C}(A) \rightarrow \mathcal{C}(B)$  taking configuration  $x$  to its direct image  $fx$  is injective. [Recall a map  $f : A \rightarrow B$  is mono iff for all maps  $g, h : C \rightarrow A$  if  $fg = fh$  then  $g = h$ .] Taking  $B$  to be the event structure comprising two concurrent events, can you find an event structure  $A$  and an example of a total map  $f : A \rightarrow B$  of event structures which is both mono and where  $f$  is not injective as a function on events?

□

*Exercise 3.* Verify that the finite configurations of an event structure form a stable family.

□

*Exercise 4.* Say an event structure  $A$  is *tree-like* when its concurrency relation is empty (so two events are either causally related or inconsistent). Suppose  $B$  is tree-like and  $f : A \rightarrow B$  is a total map of event structures. Show  $A$  must also be tree-like, and moreover that the map  $f$  is rigid, *i.e.* preserves causal dependency.

*Exercise 5.* Let  $\mathcal{F}$  be a nonempty family of sets satisfying the Completeness axiom in the definition of stable families. Show  $\mathcal{F}$  is coincidence-free iff

$$\forall x, y \in \mathcal{F}. x \not\subseteq y \implies \exists x_1, e_1. x \stackrel{e_1}{\dashv} x_1 \subseteq y.$$

[Hint: For ‘only if’ use induction on the size of  $y \setminus x$ .] □

*Exercise 6.* Prove Proposition 9: Let  $f : \mathcal{F} \rightarrow \mathcal{G}$  be a map of stable families. Let  $e, e' \in x$ , a configuration of  $\mathcal{F}$ . Show if  $f(e) \leq_{f_x} f(e')$  (with both  $f(e)$  and  $f(e')$  defined) then  $e \leq_x e'$ .

*Exercise 7.* Prove the two propositions 12 and 13. □

*Exercise 8.* (From Section 3.2) For an event structure  $E$ , show  $\mathcal{C}^\infty(E) = \mathcal{C}(E)^\infty$ . □

*Exercise 9.* (From Section 3.2) Let  $\mathcal{F}$  be a stable family. Show  $\mathcal{F}^\infty$  satisfies:

*Completeness:*  $\forall Z \subseteq \mathcal{F}^\infty. Z \uparrow \implies \bigcup Z \in \mathcal{F}^\infty$ ;

*Stability:*  $\forall Z \subseteq \mathcal{F}^\infty. Z \neq \emptyset \ \& \ Z \uparrow \implies \bigcap Z \in \mathcal{F}^\infty$ ;

*Coincidence-freeness:* For all  $x \in \mathcal{F}^\infty$ ,  $e, e' \in x$  with  $e \neq e'$ ,

$$\exists y \in \mathcal{F}^\infty. y \subseteq x \ \& \ (e \in y \iff e' \notin y);$$

*Finiteness:* For all  $x \in \mathcal{F}^\infty$ ,

$$\forall e \in x \exists y \in \mathcal{F}. e \in y \ \& \ y \subseteq x \ \& \ y \text{ is finite .}$$

Show that  $\mathcal{F}$  consists of precisely the finite sets in  $\mathcal{F}^\infty$ . □

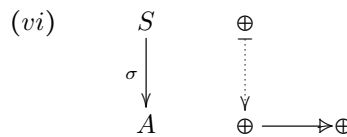
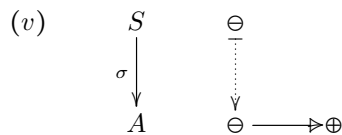
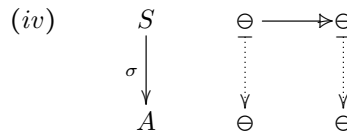
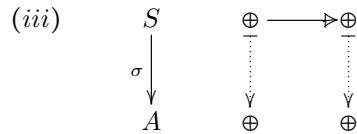
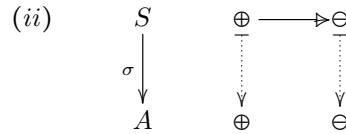
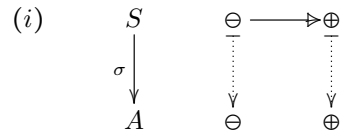
*Exercise 10.* Let  $A$  be the event structure consisting of two distinct events  $a_1 \leq a_2$  and  $B$  the event structure with a single event  $b$ . Following the method of Section 3.3.1 describe the product of event structures  $A \times B$ . □

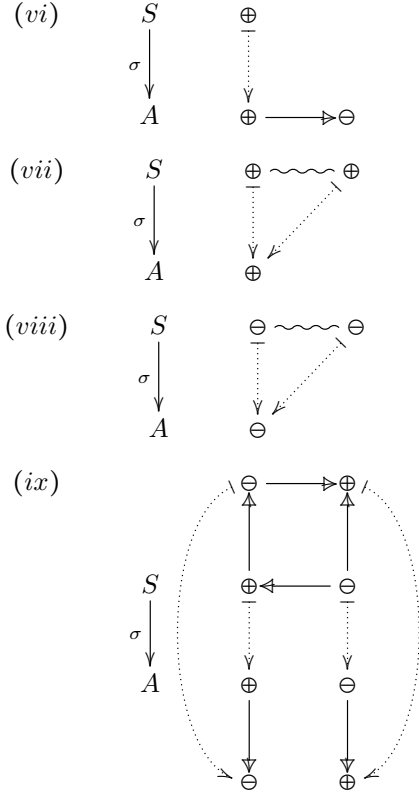
## On strategies

*Recommended exercises: 11, 12, 13, 14, 15, 17.*

*Exercise 11.* Consider the empty map of event structures with polarity  $\emptyset \rightarrow A$ . Is it a strategy? Is it a deterministic strategy? Consider now the identity map  $\text{id}_A : A \rightarrow A$  on an event structure with polarity  $A$ . Is it a strategy? Is it a deterministic strategy?  $\square$

*Exercise 12.* For each instance of total map  $\sigma$  of event structures with polarity below say whether  $\sigma$  is a strategy and whether it is deterministic. In each case give a short justification for your answer. (Immediate causal dependency within the event structures is represented by an arrow  $\rightarrow$  and inconsistency, or conflict, by a wiggly line  $\rightsquigarrow$ .)





□

*Exercise 13.* Let  $\text{id}_A : A \rightarrow A$  be the identity map of event structures, sending an event to itself. Show the identity map forms a strategy in the game  $A$ . Is it deterministic in general? □

*Exercise 14.* Show any strategy  $\sigma : A \rightarrow B$  has a dual strategy  $\sigma^\perp : B^\perp \rightarrow A^\perp$ . In more detail, supposing  $\sigma : S \rightarrow A^\perp \parallel B$  is a strategy show  $\sigma^\perp : S \rightarrow (B^\perp)^\perp \parallel A^\perp$  is a strategy where

$$\sigma^\perp(s) = \begin{cases} (1, b) & \text{if } \sigma(s) = (2, b) \\ (2, a) & \text{if } \sigma(s) = (1, a). \end{cases}$$

□

*Exercise 15.* Let  $B$  be the event structure consisting of the two concurrent events  $b_1$ , assumed  $-ve$ , and  $b_2$ , assumed  $+ve$  in  $B$ . Let the strategy  $\sigma : \emptyset \rightarrow B$  comprise the event structure  $s_1 \rightarrow s_2$  with  $s_1$   $-ve$  and  $s_2$   $+ve$ ,  $\sigma(s_1) = b_1$  and  $\sigma(s_2) = b_2$ . In  $B^\perp$  the polarities are reversed so there is a strategy  $\tau : B \rightarrow \emptyset$  comprising the event structure  $t_2 \rightarrow t_1$  with  $t_2$   $-ve$  and  $t_1$   $+ve$  yet with  $\tau(t_1) = \bar{b}_1$  and  $\tau(t_2) = \bar{b}_2$ . Describe the composition  $\tau \circ \sigma$ . □

*Exercise 16.* Say an event structure is *set-like* if its causal dependency relation is the identity relation and all pairs of distinct events are inconsistent. Let  $A$  and  $B$  be set-like event structures. In this case, can you see a simpler way to describe strategies and deterministic strategies  $A \rightarrow B$ , and what composition of strategies corresponds to?  $\square$

*Exercise 17.* By considering the game  $A$  comprising two concurrent events, one +ve and one -ve, show there is a nondeterministic pre-strategy  $\sigma : S \rightarrow A$  such that  $s \rightarrow s'$  in  $S$  without  $\sigma(s) \rightarrow \sigma(s')$ . Could you find such a counterexample were  $\sigma$  deterministic? Explain.  $\square$



# Appendix B

## Projects

*The projects are quite ambitious and to some extent open-ended. You can achieve a good grade, even in the more technical questions, without completing every part. You may use any results from the notes provided you state them clearly.*

**Project 1. Stable families with coincidence.** There are possibly good reasons to investigate event structures and stable families in which the causal dependency relation is a pre-order rather than a partial order (*cf.* the work on “round abstraction” in circuits of Ghica and Menea). In particular, investigate stable families but without the axiom of coincidence-freeness; what are their maps, what are their products, how do they relate to event structures? [My ICALP 1982 paper and report on “Event structure semantics of CCS and related languages,” available from my Cambridge homepage, might be helpful for proofs.]

**Project 2. Strategies from maps of event structures.** In this project you are guided part of the way to showing that  $f : A \rightarrow B$ , a partial map between event structures with polarity, can be regarded as a (special) strategy  $\sigma : A \multimap B$  in such a way that composition and identities are respected.

For  $f : A \rightarrow B$ , a partial map of event structures with polarity, we construct a strategy  $\sigma(f) : S \rightarrow A^\perp \parallel B$ . The event structure  $S$  is built as  $\text{Pr}(S)$  from a stable family  $\mathcal{S}$ . The family  $\mathcal{S}$  consists of subsets

$$\{1\} \times \bar{x} \cup \{2\} \times y, \text{ abbreviated to } (\bar{x}, y),$$

where  $x \in \mathcal{C}(A)$ ,  $y \in \mathcal{C}(B)$ , which satisfy

$$\begin{aligned} \bar{a} \in \bar{x} \ \& \ \text{pol}_{A^\perp}(\bar{a}) = + \implies f(a) \in y \ \text{and} \\ b \in y \ \& \ \text{pol}_B(b) = + \implies \exists a \in x. f(a) = b. \end{aligned}$$

(1) Show, for  $(\bar{x}, y) \in \mathcal{S}$ ,

(i)  $\forall x_0 \in \mathcal{C}(A). x_0 \sqsubseteq x \implies (\bar{x}_0, (fx_0) \cap y) \in \mathcal{S}$

(ii)  $\forall y_0 \in \mathcal{C}(B). y_0 \sqsubseteq y \implies (\bar{x} \cap [f^{-1}y_0], y_0) \in \mathcal{S}.$

(2) Show  $\mathcal{S}$  is a stable family.

With  $S =_{\text{def}} \text{Pr}(\mathcal{S})$ , define

$$\sigma(f)(s) = \begin{cases} \bar{a} & \text{if } \max(s) = (1, \bar{a}), \\ b & \text{if } \max(s) = (2, b). \end{cases}$$

(3) Show  $\sigma(f)$  is a total map of event structures  $\sigma(f) : S \rightarrow A^\perp \parallel B$  which respects polarity.

(4) Show  $\sigma(f)$  is a strategy  $\sigma(f) : A \dashrightarrow B.$

(5) Show, in the case where  $f$  is the identity map  $\text{id}_A : A \rightarrow A$ , that  $\sigma(\text{id}_A) = \gamma_A$ , the copy-cat strategy.

(6) Suppose now  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are maps of event structures with polarity. Can you show that  $\sigma(gf) \cong \sigma(g) \odot \sigma(f)$ ? (Hard)

(7) Is  $\sigma(f)$  always a deterministic strategy for all maps  $f$  of event structures with polarity? If not can you see what properties are required of  $f$  for  $\sigma(f)$  to be deterministic?

**Project 3. Winning strategies with neutral positions.** A natural generalisation of the games with winning conditions of Chapter 8 is to games  $(A, W, L)$  comprising an event structure with symmetry  $A$  and disjoint subsets  $W$  and  $L$  of  $\mathcal{C}^\infty(A)$  which specify the winning and losing configurations without necessarily having that one is the complement of the other—configurations in  $\mathcal{C}^\infty(A) \setminus (W \cup L)$  would be *neutral* positions. Imitate the constructions on games and winning conditions of Chapter 8 in this broader framework. Adopt the same definition of winning strategy as before. For the new dual operation and parallel composition take

$$G^\perp = (A, L_G, W_G) \text{ and } G \parallel H = (A \parallel B, W_G \parallel \mathcal{C}^\infty(B) \cup \mathcal{C}^\infty(A) \parallel W_H, L_G \parallel L_H),$$

where  $G = (A, W_G, L_G)$  and  $H = (B, W_H, L_H)$ —the notation of Chapter 8 is being used here. In the new parallel composition to win is to win in either component and to lose is to lose in both. What is the unit of  $\parallel$ ? What are the winning and losing configurations of  $G^\perp \parallel H$ ? As before, a winning strategy from  $G$  to  $H$  is a winning strategy in  $G^\perp \parallel H$ . It is important that you try to show that the composition of winning strategies is winning (follow the pattern of the proof in Chapter 8), and that for suitable games copy-cat is winning.

**Project 4. An essay on strategies in logic.** Write an essay explaining to your best friend in humanities why logicians and philosophers are interested

in games and strategies. The papers of Johan van Benthem provide a good start.

**Project 5. Games in other models.** Take a favourite model, *e.g.* transition systems, languages, some variety of Petri nets, Mazurkiewicz trace languages, and try to imitate the constructions on games there. You might find it convenient to allow “internal” events, which are neither moves of Opponent or Player, for instance in defining composition of strategies in your model.