HAND-IN EXERCISES FOR CATEGORY THEORY FOR CS
(September 2009)

PART I

Exercises 13, 14, 33 of the JvO's notes (obtainable from the course webpage).

Exercises 1.35, 3.7, 3.15, 4.16, 9.9, 9.12 of Lecture Notes in Category Theory by Caccamo and Winskel (obtainable from the course webpage)

Plus the following more advanced exercises.

Exercise A1 Define the horizontal and vertical composition of natural transformations. Establish that horizontal composition does indeed yield a natural transformation. Establish the 'interchange law' between horizontal and vertical composition of natural transformations.

Exercise A2 This exercise is on Kleisli structures, from which one can build a Kleisli category even without a monad (having a monad is a special case). *A Kleisli structure*: Assume a category $\mathcal{K}$, $\mathcal{E}$ a subclass of its objects, T an operation giving an object TA in $\mathcal{K}$ for any object A in $\mathcal{E}$, together with η_A in $\mathcal{K}(A, TA)$ for any A in $\mathcal{E}$ and an operation $(-)^\dagger : \mathcal{K}(A, TB) \to \mathcal{K}(TA, TB)$ for any A, B in $\mathcal{E}$ s.t.

$$\eta_A^\dagger = \mathrm{id}_{TA}, \qquad f^\dagger \circ \eta_A = f, \quad \text{and} \quad (g^\dagger \circ f)^\dagger = g^\dagger \circ f^\dagger.$$

Then, we can form a 'Kleisli category' $Kl(T)$ as follows: Its objects are those of $\mathcal{E}$. Its arrows $f : A \to_T B$ are maps $f : A \to TB$ in $\mathcal{K}$. Its identities at objects A are the η_A. Its composition of maps

$$f : A \to_T B \text{ and } g : B \to_T C$$

is given by $g^\dagger \circ f$.

1. Show that the 'Kleisli category' is indeed a category.
2. Let T be a monad on a category $\mathcal{K}$, with unit η and multiplication μ. Define $(-)^\dagger : \mathcal{K}(A, TB) \to \mathcal{K}(TA, TB)$ for A, B in $\mathcal{K}$ by $f^\dagger = \mu_B \circ Tf$. Show this produces a Kleisli structure—one with choice of objects $\mathcal{E}$ all the objects in $\mathcal{K}$.

[I learnt about Kleisli structures from Martin Hyland. They are useful when an operation T is like a monad but doesn't, or isn't known to, yield the same form of structure as those on which it acts. Such a situation occurs when T is the operation of forming the presheaf category over a small category—the result isn't itself small, and in the construction of certain probabilistic domains.]

Exercise A3 Let **TS** be the category of transition systems defined as follows. Its objects are transition systems $(S, i, L, tran)$ where S is a set of states with initial state $i \in S$, L is a set of labels and $tran \subseteq S \times L \times S$ is its transitions. A map from transition system $(S, i, L, tran)$ to a transition system $(S', i', L', tran')$ is a pair (σ, λ) where $\sigma : S \to S'$ is a (total) function s.t. $\sigma(i) = i'$ and $\lambda : L \rightharpoonup L'$ is a partial function s.t.

$$(s, a, s') \in tran \;\&\; \lambda(a) \text{ is defined } \Rightarrow (\sigma(s), \lambda(a), \sigma(s')) \in tran' \text{ and}$$
$$(s, a, s') \in tran \;\&\; \lambda(a) \text{ is undefined } \Rightarrow \sigma(s) = \sigma(s')$$

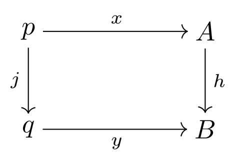
Composition is given coordinatewise.

Show that **TS** has binary products, equalisers and pullbacks. Does it have an initial object? Does it have an terminal object?

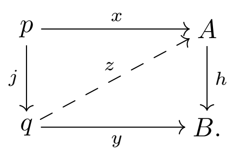
Let L be a fixed set of labels and consider the subcategory $\mathbf{TS_L}$ where all transition systems have the common label set L and maps all have the form (σ, id_L). For this subcategory, how do your answers above change? No proofs are required.

Let $\mathbf{TS_{tot}}$ be the subcategory of **TS** in which all maps have the form (σ, λ) with λ now a total function between label sets. Define a monad on $\mathbf{TS_{tot}}$ for which the Kleisli category is isomorphic to the original category **TS**. [No proof is required but your monad construction, with unit and multiplication, should be decribed clearly.]

Exercise A4 Let $\mathcal{E}$ be a category with pullbacks and products. Let $\mathbb{P}$ be a subcategory of $\mathcal{E}$. Say a map $h : A \to B$, is *open* iff for all maps $j : P \to Q$ in $\mathbb{P}$, any commuting square



can be split into two commuting triangles



Show that:

1. all isomorphisms are open;
2. the composition of open maps is open (so they form a subcategory);
3. that open maps are preserved under pullback, *i.e.* that if h is open and shares a common codomain with map f, not necessarily open, then the map obtained by pulling back h along f is open; and

4. the product of open maps is open, *i.e.* if g an h are open, then so is $g \times h$.

Say objects A and B are bisimilar if there is a span of open maps

$$A \xleftarrow{f} E \xrightarrow{g} B$$

in $\mathcal{E}$. Show this bisimilarity is an equivalence relation between objects of $\mathcal{E}$.

Characterise open maps and bisimilarity for the category of transition system $\mathbf{TS_L}$ above w.r.t. the subcategory of finite linear transition systems; a transition system is linear if it has no branching or loops.

PART II

This second part has a more open-ended character. You are asked to find ten applications of category theory in CS. Each application should be accompanied by a brief explanation, in your own words, of that application (enough that I can see you understand at least in broad terms what that application is). I guess two/three pages should be enough. A third of the marks will be assigned to this part.