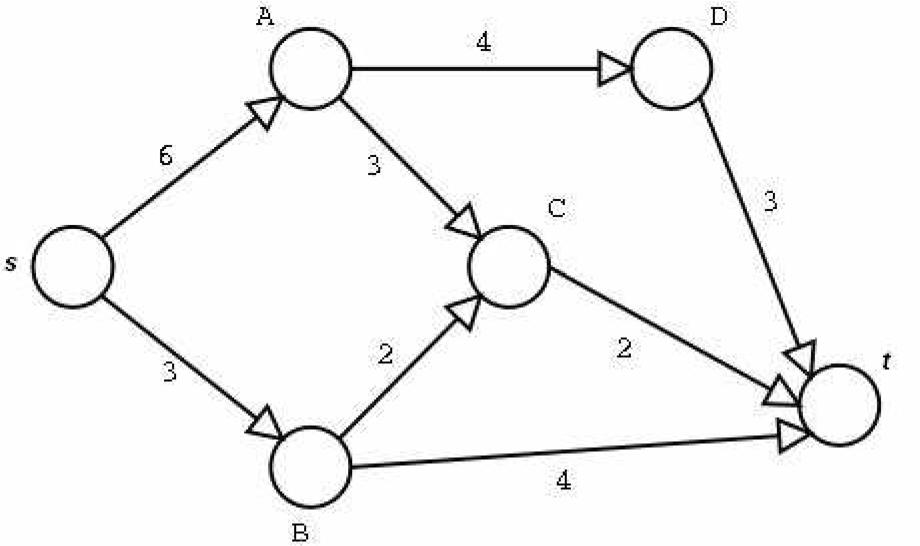
Optimization (Optimering)

Final Exam, August 17, 2005.

This problem set contains five pages, including this one.

Problem 1 (18 %)

Consider the instance of the max-flow problem shown below (capacities indicated as labels on arcs).



1A. Suppose that you run the Edmunds-Karp algorithm on this network. In the first iteration you find a path from *s* to *t* and augment along it. *Draw the resulting residual network.*

1B. What is the value of the maximum flow from *s* to *t*?

1C. State a minimum cut between s and *t* as a pair of sets of nodes.

Problem 2 (15 %)

Call an arc of a max flow instance (i.e., a flow network) *critical* if decreasing the capacity of this arc results in a decrease in the maximum flow.

2A. Give an algorithm that finds a critical arc in a given flow network *and briefly justify its correctness.*

Your algorithm should run as fast as the best available max flow algorithm.

Problem 3 (12 %)

A cargo plane can carry a maximum of 100 tons and a maximum of 60 cubic meters of cargo. There are three materials that need to be carried, and the cargo company may choose to carry any amount of each, up to the maximum amount available in each.

Material 1 has density 2 tons/cubic meter, maximum available amount 40 cubic meters, revenue $1,000 per cubic meter.

Material 2 has density 1 ton/cubic meter, maximum available amount 30 cubic meters, revenue $1,200 per cubic meter.

Material 3 has density 3 tons/cubic meter, maximum available amount 20 cubic meters, revenue $12,000 per cubic meter.

3A. Write a linear program that optimizes revenue while satisfying all the constraints.

Problem 4 (12%)

A large department store has decided to stay open for business on a continuous 24 hour basis. The store manager has divided the 24-hour day into six equal 4-hour periods and determined the following minimum personnel requirement for each period:

Time	*Personnel needed*
0:00 -- 4:00	90
4:00 -- 8:00	215
8:00 -- 12:00	250
12:00 -- 16:00	65
16:00 -- 20:00	300
20:00 -- 24:00	125

Store personnel can report for work at the beginning of any of the above time periods and will work for 8 consecutive hours. The store manager wants to know the minimum number of employees to begin work at each 4-hour segment to minimize the total number of employees.

4A. Formulate an integer linear program modelling this problem instance.

Problem 5 (12%)

Consider the following linear programming dictionary, obtained during execution of the simplex algorithm.

$x_3 = 1 + x_1 - 2x_4 + 6x_6$

$x_2 = -2x_1 - 2x_4 + 3x_6 - 2x_7$

$x_5 = 2 + 2x_1 - 8x_6 - 3x_7$

$x_8 = -3x_1 - 4x_4 - x_6 + 9x_7$

$z = 9 - 2x_1 + x_4 + 3x_6 + x_7$

5A. Is the dictionary feasible?

5B. Which variables are basic and which are non-basic?

5C. What is the basic solution associated with the dictionary?

5D. Which variables may enter the basis in the next iteration of the simplex algorithm?

5E. For *each* possible variable which may enter the basis next, state which variable(s) may leave the basis.

5F. Which pair of entering and leaving variable is specified by Bland's rule?

Problem 6 (18%)

Consider the following linear program.

Maximize $x_1 - 3x_2$ subject to

$$x_1 + x_2 \geq 2$$
$$2x_1 - x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

6A. Use the two-phase simplex algorithm to solve the linear program above. State all intermediate dictionaries.

6B. What is the dual problem?

6C. Does the dual problem have an optimal solution? If so, what is it?

Problem 7 (13%)

Given a satisfiable system of linear inequalities

$$a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1$$
$$\ldots$$
$$a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m$$

we say that an inequality is *forced-equal* in the system if for every vector **x** = $(x_1, .. x_n)$ that satisfies the system, the inequality is satisfied as an equality. (Equivalently, an inequality $\Sigma_i\, a_{ij}x_i \leq b_j$ is *not* forced-equal in the system if there is an **x** that satisfies the whole system and such that in that inequality the left-hand side is strictly smaller than the right-hand side, that is $\Sigma_i\, a_{ij}x_i < b_j$.)

For example, in

$$x_1 + x_2 \leq 2$$
$$-x_1 - x_2 \leq -2$$
$$x_1 \leq 1$$
$$-x_2 \leq 0$$

the first two inequalities are forced-equal, while the third and fourth are not forced-equal. Observe that in any satisfiable system, there is always a solution where all inequalities that are not forced-equal have a left-hand side strictly smaller than the right-hand side. In the above example, we can set $x_1 = -1$ and $x_2 = 3$, so that we have $x_1 < 1$ and $-x_2 < 0$, as well as $x_1 + x_2 = 2$ and $-x_1 - x_2 = -2$.

7A. Given a satisfiable system of linear inequalities, show how to use linear programming to determine which inequalities are forced-equal, and to find a solution where all inequalities that are not forced-equal have a left-hand side strictly smaller than the right-hand side (you may have to solve more than one linear program).