Each of the following problems will be given equal weight in the determination of the grade.

Problem 1

The Superb Soft Drink Company is producing three products: Alice's Morning Juice, Bertram's Orange Soda, and Carol's Lemonade. All products are sold to distributors in large quantities and priced by volume.

- To produce one liter of Alice's Morning Juice, the company needs 0.5 liter of orange juice, 0.2 liters of lemon juice, 6 grams of sugar, and some water.
- To produce one liter of Bertram's Orange Soda, the company needs 0.1 liters of orange juice, 30 grams of sugar, and some water.
- To produce one liter of Carol's Lemonade, the compnay needs 0.6 liters of lemon juice, 50 grams of sugar, and some water.

Alice's Morning Juice is sold at 2 dollars per liter, Bertram's Orange Soda at 1 dollar per liter, and Carol's Lemonade at 3 dollars per liter. The company can purchase sugar at 1 cent per gram, orange juice at 100 cents per liter and lemon juice at 400 cents per liter. The price of water is negligible. The company can produce and ship a combined total of at most 10000 liters of the three products in a day.

Write a linear program that expresses the problem of finding the appropriate amount of each of the products to produce daily so as to maximize profit. You do not have to solve the program.

Problem 2

Each member of a group of m celebrities wants to adopt some pets. There are n distinct pets up for adoption. After having examined the pets, each celebrity i provides a list L_i of pets she likes and also gives a lower bound l_i and an upper bound u_i on the number of pets she wants to adopt. The challenge is to allocate as many as possible of the pets to the celebrities so that each celebrity only gets pets mentioned on her list *and* a number of pets between her stated lower and upper bound.

Show how to model this problem as a network flow problem. Explicitly state the graph of the model as well as its costs, capacities and balances.

Problem 3

We are given a list of n distints computational jobs that can each be executed on one of m distinct computers. If we allocate job number i to computer number j, it will take t_{ij} seconds to execute the job. All numbers t_{ij} are known in advance. Each computer will execute the jobs we allocate to it one at a time without delay between jobs. All computers start working on the jobs we allocate to them simultaneously. We can leave office when all jobs have finished their execution.

Show how to model as an integer linear program the problem of allocating the jobs to computers in such a way that we can leave office as soon as possible. Explicitly state the decision variables, the constraints, and the objective function of your program.

Problem 4

Let

$$A = \begin{pmatrix} 1 & 0 & -2 & 1 & -2 & 2 \\ 0 & 1 & 3 & -2 & 1 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

Consider the following linear program in standard form:
Minimize $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ subject to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$, where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)'$.

Consider the basic solution $\mathbf{x}^*$ whose basis is the first two columns of A.

1. State, as a vector, the value of $\mathbf{x}^*$. Is $\mathbf{x}^*$ feasible? Is $\mathbf{x}^*$ degenerate?
2. State, for each of the non-basic variables x_3, x_4, x_5, x_6, the reduced cost of that variable.
3. If the simplex algorithm is started with $\mathbf{x}^*$ being the initial basic solution, which columns may enter the basis in the first iteration of the simplex algorithm?
4. For each of the columns that may enter the basis in the first iteration of the simplex algorithm, which columns may leave the basis?

Problem 5

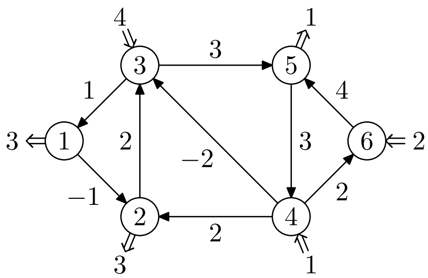
Consider the linear program given by

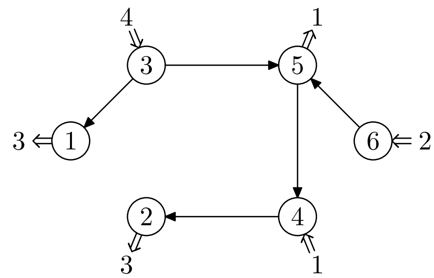
$$\begin{array}{lrcrcrcrcl} \min\limits_{x} & 15x_1 & + & 15x_2 & + & 8x_3 & + & 10x_4 & & \\ \text{s.t.} & 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & \geq & 5 \\ & 2x_1 & + & 3x_2 & + & 2x_3 & + & x_4 & \geq & 10 \\ & & & & & & & x_i & \geq & 0, \quad i = 1 \ldots 6 \end{array}$$

1. The dual program only has two variables, so the solution set can be drawn in two dimensions. Draw the solution set and state the optimal solution to the dual program.
2. Use complementary slackness to identify the optimal basis in the primal problem. State the basic variables of this optimal basis.
3. Compute the optimal solution to the primal problem and state this optimal solution.

Problem 6

Consider the uncapacitated network flow problem given by the graph on the left (labels on arcs are costs).





The tree on the right is the tree corresponding to a tree solution to the problem.

1. Compute this tree solution and present it graphically, by making a copy of the drawing of the tree, and labeling the arcs with the appropriate flow values.
2. Compute the values of the dual variables. Write these values at the appropriate vertices.
3. Compute the reduced cost of each of the four arcs *not* in the tree and list these costs.
4. Consider starting the network simplex algorithm in this tree solution. For each of the arcs e with negative reduced cost, consider letting e be thearc entering the tree in the first pivoting, and indicate which tree arc would leave the tree if e enters.

Problem 7

Given a feasible set of linear inequalities P:

$$\begin{aligned} Ax &\leq \mathbf{b} \\ \mathbf{x} &\geq 0 \end{aligned} \tag{1}$$

where A is an $m \times n$ matrix. We say that the $i'th$ inequality $\mathbf{a}_i \cdot \mathbf{x} \leq b_i$ is *satisfiable with slack* if there is a feasible solution x^* to P so that $\mathbf{a}_i \cdot \mathbf{x} < b_i$. Your task is to construct a *single* linear program P' whose optimal solution identifies which inequalilties of P are satisfiable with slack. Specifically, your program P' should have decision variables u_i, so that in any optimal solution to P', u_i takes on the value 1 if and only if the i'th inequality of P is satisfiable with slack. Note that P' should be a linear program, not a (mixed) integer linear program.