Each of the following six problems will be given equal weight in the determination of the grade.

Problem 1

A company makes two products (X and Y) using two machines (A and B). Each unit of X that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each unit of Y that is produced requires 24 minutes processing time on machine A and 33 minutes processing time on machine B.

At the start of the current week there are 30 units of X and 90 units of Y in stock. Available processing time on machine A is forecast to be 40 hours and on machine B is forecast to be 35 hours.

The demand for X in the current week is forecast to be 75 units and for Y is forecast to be 95 units. Company policy is to maximise the combined sum of the units of X and the units of Y in stock at the end of the week.

Formulate the problem of deciding how much of each product to make in the current week as a linear program with two decsion variables.

Since the linear program has two variables, the solution set can be drawn in the plane. Do this, and determine an optimal solution graphically.

Problem 2

In a Danish handball league, a tournament is being played between teams $1, 2, \ldots, N$. We are in the middle of the tournament. You want to find out if it is possible for your favorite team, team 1, to win the tournament, i.e., to end up with strictly more points than any other team.

Your input is a list L_1 with a current score of each team and a list L_2 of future matches still to be played. Each such future match is played between two teams. If one team wins a match, it gets two points while the losing team gets zero. In case of a tie, both teams get zero points.

Show how to formulate this problem as a max flow problem. You can (and probably should) appeal to the integrality theorem for flow problems.

Problem 3

Given a graph $G = (V, E)$, two vertices $s, t \in V$ and a set of pairs of vertices $C \subset V \times V$, the *shortest viable path problem* consists in finding a path containing at most one vertex from each pair of vertices in C and having the minimum possible number of edges among such paths.

Give an integer linear programming formulation of the shortest viable path problem.

Problem 4

Let

$$A = \begin{pmatrix} 1 & 2 & -1 & 2 & 2 & 0 \\ 2 & 1 & 2 & -1 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

Consider the following linear program in standard form:
Minimize $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ subject to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$, where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)'$.

Consider the basic solution $\mathbf{x}^*$ whose basis is the last two columns of A.

1. State, as a vector, $\mathbf{x}^*$. Is $\mathbf{x}^*$ feasible? Is $\mathbf{x}^*$ degenerate?
2. State, for each of the non-basic variables x_1, x_2, x_3, x_4, the reduced cost of that variable.
3. If the simplex algorithm is started with $\mathbf{x}^*$ being the initial basic solution, which columns may be the column that enters the basis in the first iteration of the simplex algorithm?

4. For each of the columns that may enter the basis in the first iteration of the simplex algorithm, which columns may be the column that leaves the basis?

Problem 5

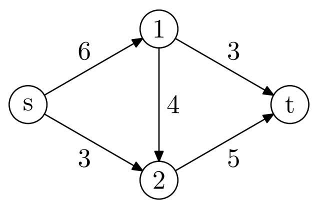
What the dual of the following linear progamming problem?

$$\begin{array}{lrcrcrcr} \min & 3x_1 & - & 5x_2 & + & 8x_3 & & \\ \text{s.t.} & x_1 & & & - & x_3 & = & 6 \\ & & & 2x_2 & - & x_3 & \geq & -5 \\ & x_1 & + & x_2 & & & \leq & 8 \\ & & & x_2 & \leq 0, & x_3 & \geq 0 & \end{array}$$

Use complementary slackness to argue that $x = (6, 0, 0)$ is an optimum solution to the program.

Problem 6

Consider the max flow instance given below (labels on arcs are capacities):



Execute the Ford-Fulkerson algorithm, with the following rule for choice of augmenting path: always choose a path along which as much as possible flow can be pushed, thereby increasing the total flow as much as possible in each single step.

For each iteration, indicate the augmenting path you use.

What is the value of the maximum flow in the network?

List, as a pair of sets, a minimum capacity cut in the network.