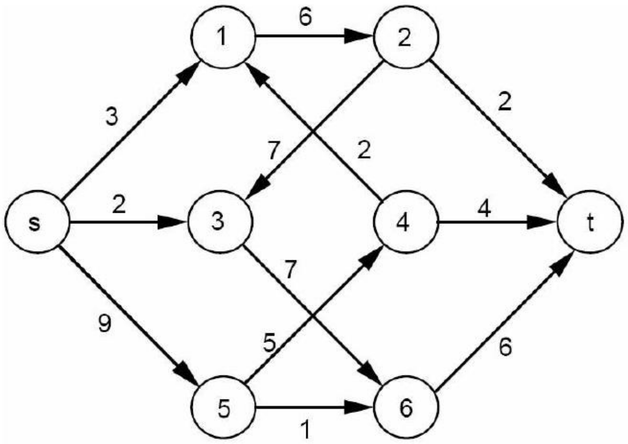
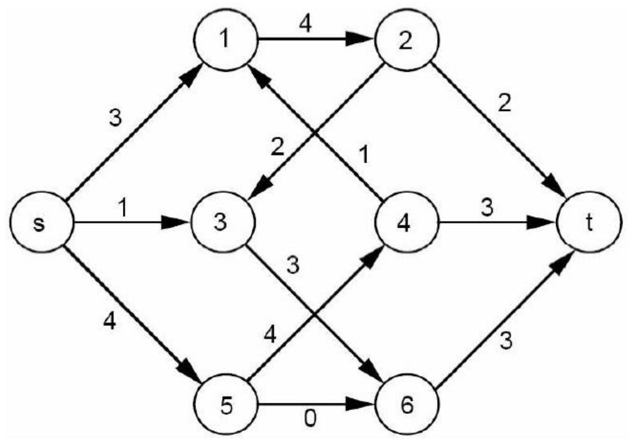
The following problems are weighted equally when determining the grade.

Problem A

In the following flow network G, the capacities $c(u, v)$ are given as edges (u, v).



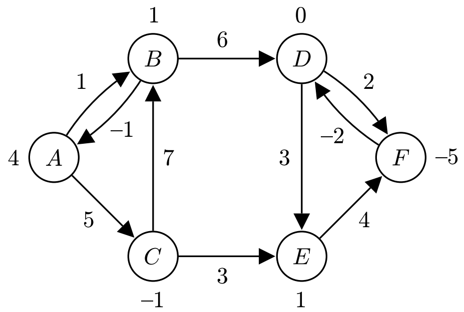
The following figure indicates a feasible flow in G. Labels at edges are now flows.



1. What is the value of the flow f?
2. Let $S = \{s, 1, 3, 5\}$ and $T = \{2, 4, 6, t\}$. What is $f(S, T)$?
3. Draw the residual network, with capacities indicated at edges. Does it have an augmenting path? If so, indicate it on your drawing.
4. Exhibit a maximum flow in G (draw it as the flow f was drawn above).
5. State, as a pair of sets, a minimum cut in G.

Problem B

The following flow network with costs G' is an instance of the min cost flow problem, with labels on edges indicating capacities and labels at vertices indicating balances (costs are not listed):



1. Find a feasible flow f' for which $f'(B, D) = 4$. Draw the flow f', by copying the network G' and labeling all edges with the flow through them.

2. Suppose all edges of G' with positive capacity have cost 1 and suppose f' happens to be the current flow at some stage in the execution of Klein's algorithm. Is there an augmenting cycle? If so, list such a cycle as an ordered sequence of vertices.

3. Suppose all edges of G' with positive capacity have cost -1 and suppose f' happens to be the current flow at some stage in the execution of Klein's algorithm. Is there an augmenting cycle? if so, list such a cycle as an ordered sequence of vertices.

Problem C

The following dictionary d was obtained directly (i.e., without pivoting) from a linear program P in standard form, by introducing slack variables x_5, x_6, x_7.

$$
\begin{array}{rcl}
x_5 &=& 6 - x_1 + x_2 - 4x_3 - x_4 \\
x_6 &=& 2 - 3x_1 - x_2 + 4x_3 + 2x_4 \\
x_7 &=& 1 - 3x_1 - x_2 - 2x_3 - x_4 \\
\hline
z &=& -x_1 + x_2 + 2x_3 + x_4
\end{array}
$$

1. Write down P as a linear program in standard form, specifying its variables, its objective function (and whether the objective function should be maximized or minimized) and its constraints.

2. Write down the dual D of P as a linear program in standard form, specifying its variables, its objective function (and whether the objective function should be maximized or minimized) and its constraints.

3. Write down the basic solution associated to the dictionary d (state it as a vector, explicitly listing all seven entries).

4. As the basic solution associated to d is feasible, we can apply the simplex algorithm directly to d. In the first iteration of the simplex algorithm we may take x_4 to be the entering variable and x_7 to be the leaving variable. However there may also be other options. List all possible pairs of entering and leaving variables for the first iteration of the simplex algorithm.

5. Assume that we actually choose x_4 to be the entering and x_7 to be the leaving variable. Which dictionary do we arrive at after pivoting? (Write down the dictionary explicitly).

6. What is an optimal solution to P? (State it as a vector, explicitly listing all four entries.)

7. What is an optimal solution to D? (State is as a vector, explicitly listing all three entries.)

Problem D

A shipping firm wants to phase out a fleet of m cargo ships over a period of p years. At the beginning of year k, $k = 1, .., p + 1$, the firm can sell any number of ships that it owns, getting a cash inflow of s_k for each of them. The shipping jobs of the firm demand that the firm is in possession of d_k ships during year k, $k = 1, .., p$. If the firm does not own sufficiently many ships to meet this demand, it must hire additional ships at the beginning of the year. The cost of hiring a ship for the duration of year k is h_k, $k = 1, .., p$. The goal is to construct a schedule for selling and hiring ships, so that the cash assets of the firm at the beginning of year $p + 1$ is maximized. Formulate this problem as a min cost flow problem.

Please state explicitly the vertices, the edges, the capacities, the costs and the balances of your min cost flow model. Also state how a solution to the original problem can be derived from the optimal solution to the min cost flow problem.

Problem E

You need to buy some filing cabinets. You know that Cabinet X costs \$10 per unit, requires six square feet of floor space, and holds eight cubic feet of files. Cabinet Y costs \$20 per unit, requires eight square feet of floor space, and holds twelve cubic feet of files. You have been given \$140 for this purchase, though you don't have to spend that much. The office has room for no more than 72 square feet of cabinets. We want to determine how many of which of the two types of filing cabinets to buy, in order to maximize storage volume.

1. Show how the problem above can be formulated as an integer linear program with two variables. Explicitly state the program.

2. Ignore the integrality constraints of the program, and draw the set of feasible solutions as a region in the plane.

3. What is the optimal solution to the original problem?

Problem F

Given a flow network G and a budget C, we consider the following *interdiction problem*: Decrease the capacities of some subset of the edges of G so that

- The total decrease in capacity is at most C. That is, if $C = 3.2$, we could, for instance, decrease the capacity of one edge by 1 and the capacity of another edge by 2.2,
- No edge gets a negative capacity, and
- the value of the maximum flow of the resulting network G' is as small as possible.

Suppose a maximum flow of G has already been computed and that the flow through all edges in this maximum flow is known to you.

1. Describe how to solve the interdiction problem in linear time in the size of G, given this information. You do not have to prove correctness of your method.