Computational Game Theory, Fall 2009	November 27
Lecture 8: Minimax values and strategies in 3-player games	
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1 Three-player games

Definition 1 In a strategic form game we have:

- A set I of players, $I = \{1, 2, \ldots, l\}$.
- For each player i, a set of pure strategies S_i.
- For each player i, a set of mixed strategies $\tilde{S}_i = \Delta(S_i)$, which is a probability distribution over pure stategies.
- For each player i, an expected utility $\tilde{u}_i : \tilde{S}_1 \times \tilde{S}_2 \times \cdots \times \tilde{S}_l \to \mathbb{R}$.

So far we have only considered two-player games, that is $|I| = 2$. In this lecture we will consider three-player games. For two-player, 0-sum games we have the security (lower) value

$$\underline{v} = \max_{\sigma_1 \in \tilde{S}_1} \min_{\sigma_2 \in \tilde{S}_2} \tilde{u}_1(\sigma_1, \sigma_2)$$

which is the gauranteed payoff for player 1. In fact, we can assume that Player 2 plays a pure strategy as a response to Player 1, so we also have

$$\underline{v} = \max_{\sigma_1 \in \tilde{S}_1} \min_{s_2 \in S_2} \tilde{u}_1(\sigma_1, s_2)$$

We also have the threat (upper) value

$$\bar{v} = \min_{\sigma_2 \in \tilde{S}_2} \max_{s_1 \in S_1} \tilde{u}_1(s_1, \sigma_2)$$

.

Von Neumann's min-max theorem gives that $\underline{v} = \bar{v}$. In this lecture we will consider three player games, that is, we assume that $|I| = 3$. And we will try to find out whether Von Neumann's theorem also holds for these games. For a three player game we have

$$\underline{v} = \max_{\sigma_1 \in \tilde{S}_1} \min_{\substack{s_2 \in S_2 \\ s_3 \in S_3}} \tilde{u}_1(\sigma_1, s_2, s_3)$$

$$\bar{v} = \min_{\substack{\sigma_2 \in \tilde{S}_2 \\ \sigma_3 \in \tilde{S}_3}} \max_{s_1 \in S_1} \tilde{u}_1(s_1, \sigma_2, \sigma_3)$$

.

Think of a three-player game as two players (Players 2 and 3) against one (Player 1). For example Player 2 and 3 could be a married couple and hence have a joint utility function. Then we can rewrite the security and threat values as described in the following sections.

2 Rewriting the security value

$$\underline{v} = \max_{\sigma_1 \in \tilde{S}_1} \min_{(s_2,s_3)\in S_2 \times S_3} \tilde{u}_1^*(\sigma_1, (s_2, s_3))$$

where $\tilde{u}_1^*$ is the utility function for player 1 in the corresponding two player game and the strategy space of the game is $S_1 \times (S_2 \times S_3)$. In this case we can apply Von Neumann's min-max theorem as we know it for two player games, as the expression is exactly the same as it would be if Player 2 and Player 3 is really one player with strategy space $S_2 \times S_3$. Hence we have the second equallity below

$$\begin{aligned} \underline{v} &= \max_{\sigma_1 \in \tilde{S}_1} \min_{(s_2,s_3)\in S_2 \times S_3} \tilde{u}_1^*(\sigma_1, (s_2, s_3)) \\ &= \min_{\sigma_{2,3} \in \Delta(S_2 \times S_3)} \max_{s_1 \in S_1} \tilde{u}_1^*(\sigma_1, \sigma_{2,3}) \end{aligned}$$

Note that there is a correlation between the strategies of player 2 and 3. This is represented by the notation $\sigma_{2,3}$. When strategies are uncorrelated players assign probabilities to each row or columns in the matrix of the game. When strategies are correlated they assign probabilities to each cell in the matrix.

The maximin value of three player games can be computed in polynomial time using linear programming.

3 Rewriting the threat value

Here we cannot really do anything. So we still have

$$\bar{v} = \min_{\substack{\sigma_2 \in \tilde{S}_2 \\ \sigma_3 \in \tilde{S}_3}} \max_{s_1 \in S_1} \tilde{u}_1^*(s_1, \sigma_2, \sigma_3)$$

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Example: Consider a $2 \times 2 \times 2$ game with $\tilde{u}_1(s_i, s_j, s_k) = A_i(j.k)$. Assume that

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

Show that $\underline{v} < \bar{v}$.

We have

$$\begin{aligned} \underline{v} &= \max_{\sigma_1 \in \tilde{S}_1} \min_{\substack{s_2 \in S_2 \\ s_3 \in S_3}} \tilde{u}_1^*(\sigma_1, s_2, s_3) \\ &= \max_{p \in [0,1]} \min_{\substack{s_2 \in S_2 \\ s_3 \in S_3}} (-p, -2(1-p)) \end{aligned}$$

where the second equality arises when player 1 chooses the p such that the two expressions that player 2 and 3 chooses the minimum of are equal. That is, $-p = -2(1-p) \quad \Rightarrow \quad p = \frac{2}{3}$. Let now

$\sigma_2 = \begin{pmatrix} q \\ 1-q \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} r \\ 1-r \end{pmatrix}$. With this notation we can write the threat value as

$$\begin{aligned}
\bar{v} &= \min_{\substack{\sigma_2 \in \tilde{S}_2 \\ \sigma_3 \in \tilde{S}_3}} \max_{s_1 \in S_1} \tilde{u}_1(s_1, \sigma_2, \sigma_3) \\
&= \min_{q,r \in [0,1]} \max \big(-qr, -2(1-q)(1-r) \big) \\
&= \frac{-2q + 2q^2}{2-q} \\
&= -6 + 4\sqrt{2} \quad \approx \quad -0,343
\end{aligned}$$

and the second equality above arises when we maximize for $-qr = -2(1-q)(1-r) \quad \Rightarrow \quad r = \frac{2-2q}{2-q}$. By finding the derivative, setting that equal to zero, and solving we obtain the third equality above. Hence we have that $\bar{v} \approx -0,343 > -\frac{2}{3} = \underline{v}$. We have now shown that the min-max theorem in general is not true for three-player games. This means that three-player games are more complicated than two-player games. From the last section, we see that we can think of the difference between the lower and the upper value as the amount the married couple gains if the husband and wife are allowed to correlate their mixed strategies (say, by having access to a joint source of randomness).

4 Parameterized problems

Now we will consider the complexity of computing the minimax value $\bar{v}$. We shall be interested in the case where the number of strategies of Player 1 is a parameter k. Let us start with some definitions.

Definition 2 A parameterized problem is a language L in which each instance is associated with some parameter $k \in \mathbb{N}$ (e.g. k-Clique, k-NodeCover).

Note that given a graph $G = (V, E)$ a k-NodeCover is a set of k vertices W such that $\forall (u,v) \in E : u \in W \vee v \in W$.

Definition 3 We say that L is fixed parameter tractable if it can be solved in time at most some function of the form $f(k) \cdot n^c$ where $f(k)$ is a function of the parameter k, n is the input size in bits, and c is a constant, not depending on k. We denote the class of fixed parameter tractable problems FPT.

We will show that the problem of finding the max-min value is NP-hard and also not fixed parameter tractable under some reasonable assumption when the parameter k is the number of strategies of Player 1. But first, we look at a non-game theoretic examples that shows that while NodeCover and Clique are both NP-hard, they are different from the point of view of fixed paramter tractability.

Example: We want to show that k-NodeCover $\in$ FPT. We note that the k-NodeCover problem is: Given a graph $G = (V, E)$ and k does a k-NodeCover exist?

Trivial algorithm: Try all W of size k. Computational time of this algorithm is $O\big((n \cdot m) n^k\big)$. However, we can do better. Instead of k in the exponent we would like to have an independent constant c.

Observation 1: $deg(v) > k \Rightarrow v \in W$ so we have to include v in W. Add all such v to the "NodeCover" W and remove them from G. Remove also vertices with degree 0. Let us denote the remaining subgraph og G by $G' = (V', E')$ and let n be the number of vertices in the original graph G and m the number of edges in G.

Case 1: $|E'| > k^2$

Case 2: $|E'| \leq k^2 : |V'| \leq 2k^2$. Apply the trivial algorithm. Now with complexity $O\big(k^2 \cdot k^2 \cdot (k^2)^k\big)$.

The overall complexity of the algorithm is $O(k^{2k+4} \cdot n^2)$, where n is the number of vertices. We have now seperated k from the number of vertices/input size. Hence, k-NodeCover $\in$ FPT.

Is the same true for k-Clique? We do not think so. It would be very interesting to show that we are right, assuming only P not equal to NP. We do not know how to do this, but we can show that k-Clique is not in FPT under a stronger assumption.

Definition 4 *The exponential time hypothesis (ETH) is the following statement. 3-SAT (3-satisfiability) cannot be solved in (subexponential) time $2^{o(n)}$, where n is the input size.*

ETH is obviously a stronger statement than $P \neq NP$. Chen et al. showed that ETH implies that the k-Clique cannot be solved in time $n^o(k)$. In particular, ETH implies that the k-Clique $\notin$ FTP.

5 NP-hardness and fixed parameter intractibility of minimax values

Borgs *et al.* showed that approximating minimax values in 3-player games is NP-hard. The following theorem shows this and also fixed parameter intractibility.

Theorem 5 (Hansen et al.) *Deciding k-Clique for a graph with n vertices polynomial time reduces to approximating the minimax value $\bar{v}$ of a $2k \times kn \times kn$ game with 0-1 payoffs within $\frac{1}{4k^2}$. In particular, approximating $\bar{v}$ is NP-hard and also, if we assume ETH, fixed parameter intractable when the parameter is the number of strategies of Player 1.*

Proof [sketch] Given a graph $G = (V, E)$ we want to find a k-Clique. We do that by constructing a three player game , informally, as follows

- Players 2 and 3 each pick a vertex of G and a label (out of n vertices and k labels).
- Player 1 pick a player and a label.
- Player 1 wins if he guessed correctly or the choice of 2 and 3 is inconsistent with a labeled clique.

By inconsistency with a labeled clique we mean that there player 2 and player 3 choose different vertices with no edge between them or with the same label *or* that they chose the same vertex but with different labels. Formally, the strategy spaces of the players are $S_1 = \{2, 3\} \times \{1, \ldots, k\}$, $S_2 = V \times \{1, \ldots, k\}$, and $S_3 = V \times \{1, \ldots, k\}$ respectively. The utility function for player 1 is

$$u_1\left((i, l_1), (v_2, l_2), (v_3, l_3)\right) = \begin{cases} 1 & if\, l_1 = l_i \\ 1 & if\, v_2 = v_3 \wedge l_2 \neq l_3 \\ 1 & if\, v_2 \neq v_3 \wedge l_2 = l_3 \\ 1 & if\, v_2 \neq v_3 \wedge (v_2, v_3) \notin E \\ 0 & otherwise \end{cases}$$

The labelling should be unique, that is, player 2 and 3 has to agree on labelling. We have two cases: Either G contains a clique or not.

Suppose G contains a k-Clique. Then $\bar{v} \leq \frac{1}{k}$. Suppose now that G does not contain a k-Clique. Then we will show that $\bar{v} \geq \frac{1}{k} + \frac{1}{4k^2}$.

Ideas:

- Players 2 and 3 are forced to ude all labels. They cannot assign two much to any label.
- Show that if player 2 and player 3 pick the same label, then with high probability they also pick the same vertex.
- There is some set of vertices that player 2 and player 3 pick with high probability. Call this set of vertices $C \subseteq V$ (with $|C| \leq k$).
- Either $|C| = k$ and one edge is missing or $|C| < k$.

□

We can easily improve $\frac{1}{4k^2}$. In fact, we can show that for any $\epsilon > 0$ it is NP-hard to approximate the minimax value of an $n \times n \times n$ game within $\frac{1}{n^{\varepsilon}}$, i.e, to $\epsilon \log n$ digits of accuracy. To prove this, we use a trick called *padding*. From the above, we know that k-CLIQUE reduces to approximating $\bar{v}$ of a $2k \times kn \times kn$ game within $\frac{1}{4k^2}$. Also, for any $\epsilon > 0$, when $k = n^{\epsilon}$, k-CLIQUE is NP-hard. Let $\varepsilon > 0$ be given and let $k = n^{\epsilon}$. Given an instance of k-CLIQUE, first apply the above reduction. Then make *copies* of the strategies the Player 1 so that the $2k \times kn \times kn$ game is converted into an $n' \times n' \times n'$ game, where $n' = kn = n^{1+\epsilon}$. We conclude that it is NP-hard to approximate the upper value of an $n' \times n' \times n'$ game within $1/4(n^{\epsilon})$. Since ϵ is arbitrary, we in fact have that it is NP-hard to approximate the minimax value of an $n \times n \times n$ game within $1/n^{\epsilon}$.

There is an open problem: Can one approximate the minimax value of a three-player game with 0-1 payoffs within some constant $\varepsilon > 0$ (say, $\epsilon = 0.01$, i.e. with two decimal digits of accuracy) in polynomial time? We do not know, but there is an algorithm which is almost, but not quite polynomial.

Theorem 6 *The minimax value $\bar{v}$ of an $n \times n \times n$ game with payoffs between 0 and 1 can be approximated within $\varepsilon > 0$ in time $n^{O(\frac{\log n}{\varepsilon^2})}$, i.e, in quasi-polynomial time.*

This theorem implies that the problem is not NP-hard unless NP $\leq$ QuasiP, violating ETH. That is, it is one of those annonying problems which we do not know how to put in P but which we do not believe to be NP-hard either. We prove the theorem later, when we discuss a similar algorithm for finding approximate Nash equilibria.