Lower Bounds for External Memory Dictionaries

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BRICS
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Work presented at SODA, Jan 2003
Dictionary

- **Queries**
  - membership
  - predecessor / successor
  - range queries . . .

- **Updates**
  - insertions
  - deletions
Dictionary

- **Queries**
  - membership
  - predecessor / successor
  - range queries . . .

- **Updates**
  - insertions
  - deletions

This talk: Comparison based, membership, insertions

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Dictionaries – Comparison Based

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Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
### Dictionaries – Comparison Based

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**Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries**
## Dictionaries – Comparison Based

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<td>Borodin et al. 1982</td>
<td>$O(t)$</td>
<td>$N/2^{O(t)}$</td>
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![Graph showing search and insert times](image-url)
External Memory Model

\[ N = \text{problem size} \]
\[ M = \text{memory size} \]
\[ B = \text{I/O block size} \]

- One I/O moves \( B \) consecutive records from/to disk
- **Cost**: number of I/Os
- Elements can be copied and compared in internal memory

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Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
B-trees
– An External Memory Dictionary

Bayer and McCreight 1972

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
B-trees
– An External Memory Dictionary

Bayer and McCreight 1972

\[ O\left(\log_B M\right) \]

\[ O\left(\log_B \frac{N}{M}\right) \]

\[ O(B) \]
B-trees
– An External Memory Dictionary
Bayer and McCreight 1972

\[ O \left( \log_B \frac{N}{M} \right) \]

\[ O \left( \log_B M \right) \]

Search/update path

Insert
Search

\[ O \left( \log_B \frac{N}{M} \right) \text{ I/Os} \]

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
# Dictionaries – External Memory

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Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Comparisons vs. I/Os

\[ \text{Search} \]
\[ \log N \]
\[ \log N \]
\[ \text{Insert} \]

\[ \text{Search} \]
\[ \log_B \frac{N}{M} \]
\[ \log_B \frac{N}{M} \]
\[ \text{Insert} \]

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Comparisons vs. I/Os

**Comparisons**

\[ \Theta(N \log N) \]

**I/Os**

\[ \Theta\left(\frac{N}{B} \log_{M/B} \frac{N}{M}\right) \]

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**Aggarwal and Vitter 1988**
Comparisons vs. I/Os

Comparisons: $\Theta(N \log N)$
I/Os: $\Theta(\frac{N}{B} \log_{M/B} \frac{N}{M})$

Aggarwal and Vitter 1988

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Results

\[ \frac{N}{M \cdot \left( \frac{M}{B} \right)^{\Theta(\delta)}} \]

\[ \Theta(\log_\delta \frac{N}{M}) \]

\[ \delta = \text{number of I/Os for } B \text{ insertions} \]

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Results

$N/(M \cdot (\frac{M}{B})^{\Theta(\delta)})$

$\Theta(\log_\delta \frac{N}{M})$

$\frac{1}{\varepsilon} \log_B \frac{N}{M}$

$\frac{B^\varepsilon}{\varepsilon} \log B \frac{N}{M}$

$\delta = \text{number of I/Os for } B \text{ insertions}$

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Results

\[\frac{N}{(M \cdot \frac{M}{B})^{\Theta(\delta)}}\]

\[\Theta(\log_\delta \frac{N}{M})\]

\[\frac{1}{\varepsilon} \log_B \frac{N}{M}\]

\[\log_B \frac{N}{M}\]

\[\Theta(\log_{M/B} \frac{N}{M})\]

\[\log^{1+\varepsilon} N\]

\[\frac{B^\varepsilon}{\varepsilon} \log_B \frac{N}{M}\]

\[B \log_B \frac{N}{M}\]

\[B \log_B \frac{N}{M}\]

\[\delta = \text{number of I/Os for } B \text{ insertions}\]

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Buffered B-trees
– how to speedup B-tree updates by a factor $\varepsilon B^{1-\varepsilon}$

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Buffered B-trees
– how to speedup B-tree updates by a factor $\varepsilon B^{1-\varepsilon}$

- B-tree with degree $\Theta(B^\varepsilon)$

---

Searches

- $O\left(\frac{1}{\varepsilon} \log_B \frac{N}{M}\right)$
- $B$ insertions
- $O\left(\frac{B^\varepsilon}{\varepsilon} \log_B \frac{N}{M}\right)$

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Buffered B-trees
– how to speedup B-tree updates by a factor $\varepsilon B^{1-\varepsilon}$

- B-tree with degree $\Theta(B^\varepsilon)$
- Buffers of $O(B)$ delayed insertions

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Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
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– how to speedup B-tree updates by a factor $\varepsilon B^{1-\varepsilon}$

- B-tree with degree $\Theta(B^\varepsilon)$
- Buffers of $O(B)$ delayed insertions
- On buffer overflow move $O(B^{1-\varepsilon})$ elements to a child with one I/O

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Lower Bound
– optimality of buffered B-trees

- Adversary online constructs $S_1, \ldots, S_K$
- Goal: an $i$ such that $x_{ij}$ has not been in internal memory since end of $S_j$, for all $j = 1, \ldots, K$
- $x_{i1}, \ldots, x_{iK}$ form an antichain, i.e. search requires $\geq K$ I/Os
Lower Bound (Cont.)

- Let \( \hat{I} \) be the indexes \( i \) where
  - \( x_{ij} \in S_j \) but is not in internal memory at end of \( S_j \)
  - \( x_{i1}, \ldots, x_{i(j-1)} \) have not been read into internal memory by the \( \delta \frac{|S_j|}{B} \) I/Os during the insertion of \( S_j \)

- Construct \( I \subseteq \hat{I} \) such that all blocks in external memory contain \( O\left(\frac{B}{\delta}\right) \) elements \( x_{ij} \) where \( i \in I \)
  - Existence by randomized sampling with probability \( O(1/\delta) \) and Chernoff bounds, provided \( B/\delta = \Omega(\log N) \)

- Let \( x_{i(j+1)} \in S_{j+1} \) iff \( i \in I \)

\[ K = \Theta(\log_\delta \frac{N}{M}) \]
Lower Bound — Below Sorting

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<td>( \frac{N}{M \cdot \left( \frac{M}{B} \right)^{\Theta(\delta)}} )</td>
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w.l.o.g. memory and each block totally ordered after each I/O

Antichain of size \( (Borodin et al. 1982/Dillworth's lemma) \)

\[
N \leq M \cdot \left( \frac{M}{B} \right)^{\Theta(\delta)}
\]
Lower Bound — Below Sorting

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<td>$\frac{N}{M \cdot \left(\frac{M}{B}\right)^{\Theta(\delta)}}$</td>
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<td>$B$</td>
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- W.l.o.g. memory and each block totally ordered after each I/O

$$N \log M + \delta \frac{N}{B} \left( B \log \frac{M}{B} \right)$$ comparisons

- Insert in internal memory
- Merging a block with internal memory
Lower Bound — Below Sorting

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- W.l.o.g. memory and each block totally ordered after each I/O

\[
N \log M + \delta \frac{N}{B} \left( B \log \frac{M}{B} \right) \quad \text{comparisons}
\]

- Insert in internal memory
- Merging a block with internal memory

- Antichain of size (Borodin et al. 1982 / Dillworth’s lemma)

\[
\frac{N}{2^{\log M + \delta \log \frac{M}{B}}} = \frac{N}{M \cdot \left(\frac{M}{B}\right)^{\delta}}
\]

of which all elements except one are in distinct blocks

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Conclusion

\[
\frac{1}{\epsilon} \log_B \frac{N}{M} \quad \frac{B^\epsilon}{\epsilon} \log_B \frac{N}{M} \quad \Theta(\log_\delta \frac{N}{M}) \quad \frac{N}{(M \cdot \left(\frac{M}{B}\right)^\Theta(\delta))}
\]

Search

Insert

B-trees

Buffered B-trees

Sorting Threshold

Brodal, Fagerberg: Lower Bounds for External Memory Dictionaries
Conclusion

$N/(M \cdot (M/B)^{\Theta(\delta)})$

$\Theta(\log_\delta \frac{N}{M})$

$\frac{1}{\epsilon} \log_B \frac{N}{M}$

$\log_B \frac{N}{M}$

$\Theta(\log_{M/B} \frac{N}{M})$

$\log^{1+\epsilon} N$

$B / \log^3 N$

$B \log_B \frac{N}{M}$

Sorted Threshold

Buffered B-trees

B-trees

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